Topology I

Resit Exam
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1. Let V be a vector space over $\mathbb{R}$. A subset X of $\boldsymbol{V}$ is called **convex** if for any A and B in X, the segment $AB = \{tA + sB : s + t = 1\}$ is a subset of X**.**

1a. Show that any subset X of V is contained in a smallest convex subset $C(X)$ of V (called the convex hull of X). (5 pts.)

1b. Let $A_i = (0, ..., 0, 1, 0, ..., 0) \in \mathbf{R}^n$. What is the convex hull of $\{A_1, ..., A_n\}$? (3 pts.)

2. Let (X, d) a metric space. Given $a \in X$ and $\varnothing \neq B \subseteq X$, let

$$d(a, B) = \inf\{d(a, b) : b \in B\}.$$

2a. Why does $d(a, B)$ exist for all B ($\neq \varnothing$)? (2 pts.)

2b. Show that $\{a \in X : d(a, B) = 0\}$ is the closure of B. (5 pts.)

2c. Find an example of a metric space X and a nonempty closed subset $B \subseteq X$ such that for all $b \in B$, $d(a, B) < d(a, b)$. (5 pts.)

3. Let S be the set of all sequences of natural numbers (= the set of all functions from **N** into **N**). For $f = (f_0, f_1, f_2, ...)$ and $g = (g_0, g_1, g_2, ...) \in$ S, define $d(f, g) = 1/2^n$ where n is the first integer such that $f_n \neq g_n$. (If there is no such n then $f = g$ and $d(f, g)$ is defined to be 0).

3a. Show that $d(f, g) \leq \max(d(f, h), d(h, g))$ and that the equality holds in case $d(f, h) \neq d(h, g)$. (5 pts.)

3b. Show that (S, d) is a metric space. (3 pts.)

3c. Let $f \in S$. Find the open ball of center f and radius 1. (3 pts.)

3d. How many open balls are there in S? (5 pts.)

3e. Show that S is not compact. (5 pts.)

3f. Show that every open (resp. closed) ball of S is also closed (resp. open). (5 pts.)

3g. Show that every open (resp. closed) subset of S is closed (resp. open). (3 pts.)

3h. Let $\varphi_i = (\delta_{in})_n$. Show that $(\varphi_i)_i$ is a Cauchy sequence. Does it have a limit? (8 pts.)

3i. Is S a complete metric space? (10 pts.)

3j. Consider the set S_o of all sequences of 0's and 1's. Note that $S_o \subseteq S$. Show that S_o is a closed subset of S. (7 pts.)

4. Recall that a topological space X is **connected** if it is not the union of the disjoint nonempty open subsets. Let X be a topological space.

4a. Let $A \subseteq X$ be a connected subspace of X. Show that $\overline{A}$ is connected. (5 pts.)

4b. Show that the relation "$x \equiv y$ iff x and y belong to a connected subspace of X" is an equivalence relation on X. (5 pts.)

4c. Show that each equivalence class for the above relation is a maximal connected subspace (5 pts.) and is clopen (5 pts.). Each equivalence class is called a **connected component** of X.

A **topological group** G is both a Hausdorff topological space and a group such that the multiplication map $m : G \times G \to G$ and the inversion map $i : G \to G$ given by $m(x, y) = xy$ and $i(x) = x^{-1}$ are continuous.

4d. Show that in a topological group, the connected component of 1 is a closed normal subgroup of G. (6 pts.)