Topology I

Final Exam
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Part I. General Theory (30 pts.)

1. *Show that every open subset of* $\mathbb{R}$ *is a countable union of open and disjoint intervals.*

Let U be an open subset of $\mathbb{R}$. For $x, y \in U$, set $x \equiv y$ iff $[x, y] \cup [y, x] \subseteq U$. It can be easily checked that this is an equivalence relation. The equivalence classes (which form a partition of U) must be open intervals (because U is open). Since each open interval can be coded by one of the rational numbers that it contains and since there are countably many rational numbers, there are countably many equivalence classes.

2. *Show that a closed subspace of a compact space is compact.*

Let X be a compact space and C a closed subset of X. Let $(U_i)_i$ be an open covering of C. Then $U_i = V_i \cap C$ for some open subset of X. Now the family of sets $(V_i)_i$ together with $X \setminus C$ form an open cover of X. Since X is compact, finitely many of them, say $V_1, ..., V_n, X \setminus C$, cover X. Taking the intersection of this cover with C, we see that $U_1, ..., U_n$ cover C.

3. *Show that a compact subset of a Hausdorff space is closed.*

Let X be a Hausdorff space and C a compact subset of X. Let $x \in X \setminus C$. Since X is Hausdorff, for any $c \in C$ there are disjoint open subsets U_c and V_c such that $c \in U_c$ and $x \in V_c$. Since the subsets $(U_c)_{c\in C}$ cover C and since C is compact, finitely many of the U_c, say $U_1, ..., U_n$ cover C. Now $V := V_1 \cap ... \cap V_n$, where V_i is the set corresponding to U_i, is an open set containing x. Since $V \subseteq V_i$ and $V_i \cap U_i = \emptyset$, we also have $V \cap U_i = \emptyset$. This shows that $V \cap C = \emptyset$. Thus V is an open subset of X containing x and not intersecting C. Since x is arbitrary in $X \setminus C$, this shows that $X \setminus C$ is open, i.e. that C is closed.

4. *Show that the continuous image of a compact space is compact.*

Let X and Y be topological spaces. Assume X is compact. Let $f : X \to Y$ be a continuous map. Let $(V_i)_i$ be an open cover of $f(X)$. Since f is continuous, $f^{-1}(U_i)$ is open for each i. Therefore $(f^{-1}(U_i))_i$ is an open cover of X. Since X is compact, finitely many of them suffices to cover X, say X is covered by $f^{-1}(U_1), ..., f^{-1}(U_n)$.

$$f(X) = f(f^{-1}(U_1) \cup ... \cup f^{-1}(U_n)) = f(f^{-1}(U_1)) \cup ... \cup f(f^{-1}(U_n)) \subseteq U_1 \cup ... \cup U_n$$

Thus $f(X)$ is compact.

5. *Let X and Y be subsets of a topological space and endow each of X, Y and X ∪ Y with the induced topology. Let Z be another topological space. Let f : X → Z and g : Y → Z be two continuous maps which agree on X ∩ Y. Define f ∪ g : X ∪ Y → Z by glueing f and g. Give an example where f ∪ g is not continuous. Show that if both X and Y are closed (or open) in X ∪ Y then f ∪ g is continuous.*

Example: Let $X = (0, 1)$ and $Y = [1, 2)$, both viewed in $\mathbf{R}$. Let $Z = \mathbf{R}$ and $f(x) = 0$ for $x \in X$ and $g(y) = 1$ for all $y \in Y$. Then f and g are continuous and since $X \cap Y = \emptyset$, f and g agree on $X \cap Y$. Clearly $f \cup g : (0, 2) \to \mathbf{R}$ is not continuous, since it takes only two values 0 and 1.

Assume X and Y are both open in $X \cup Y$. Let U be an open subset of Z. Then $(f \cup g)^{-1}(U) = f^{-1}(f(X) \cap U) \cup g^{-1}(g(X) \cap U)$. Since f and g are continuous, the two constituants of the union are open in X and Y respectively. Since X and Y are open in $X \cup Y$, $(f \cup g)^{-1}(U)$ is open in $X \cup Y$. This shows taht $f \cup g$ is continuous.
In case X and Y are both open in $X \cup Y$, instead of taking an open subset U of z, take a closed subset C of Z and repeat the above argument replacing the word "open" by the word "closed".

Part II. Identification Space (30 pts.)
Let X be a topological space, Y a set and $\pi : X \rightarrow Y$ a map from X into Y.
6. *Show that if we endow Y with the weakest topology, then π is continuous.*
The only open subsets of Y in the weakest topology are $\emptyset$ and Y and their inverse image is $\emptyset$ and X respectively, which are both open in X. Thus π is continuous.

7. *Show that there is a unique richest topology on Y that makes π continuous.*
Since "taking the π^{-1}" respects arbitrary unions and intersections, if π is continuous in the topologies τ_1 and τ_2 on Y, π remains continuous when Y is endowed with the topology generated by the open subsets of τ_1 and τ_2. This fact generalizes to an arbitrary number of topologies very easily. More precisely, let $T = \{\tau : \tau$ is a topology on Y and π is continuous in that topology$\}$. Now consider the topology σ generated by $\{V \subseteq Y : V$ is open for one of the topologies in $T\}$. Now π is continuous in this topology and this topology is certainly the largest on Y that makes π continuous.
There is a more explicit way of finding this topology: Call a subset V of Y open if $\pi^{-1}(V)$ is open in X. This certainly defines a topology and it can only be the largest that makes π continuous.

8. *Suppose $X = \mathbf{R}$ with the usual topology and $Y = \mathbf{R}$. What is the richest topology on Y that makes the map $\pi : X \rightarrow Y$ defined by $\pi(x) = x^2$ continuous? The same question with $\pi(x) = x^3$.*
Proceeding as in the second paragraph of number 7, we see that the open subsets of the maximal topology that makes the squaring map continuous are the unions of open subsets of $\mathbf{R}^{\geq 0}$ and of arbitrary subsets of $\mathbf{R}^{<0}$. For the cubing map, we find the usual topology on $\mathbf{R}$.

One considers most often the above topology on Y only when the map π is onto. In that case the set Y can be regarded as the quotient set $X/\equiv$ where $\equiv$ is the equivalence relation defined by "$x_1 \equiv x_2$ iff $\pi(x_1) = \pi(x_2)$", i.e. Y can be viewed as the set X where the points of each equivalence class are identified with each other. For that reason Y is called an **identification space**, π the **identification map** or the **projection** and the richest topology on Y that makes f continuous the **identification topology**. For example, the cylinder without the top and the bottom, the Möbius strip and the torus are all identification spaces of the rectangle $[0, 1]^2$.
From now on, we will assume that the map $\pi : X \rightarrow Y$ is onto.

9. *Let Y be an identification space for X (with respect to some $\pi : X \rightarrow Y$). Let Z be a topological space and $f : Y \rightarrow Z$ be a map. Show that a f is continuous iff $f \circ \pi : X \rightarrow Z$ is continuous.*

If f is continuous then $f \circ \pi$ is certainly continuous, being the composition of two continuous maps. Conversely, assume that $f \circ \pi$ is continuous. Let W be an open subset of

Z. By hypothesis $(f \circ \pi)^{-1}(W)$ is open, i.e. $\pi^{-1}(f^{-1}(W))$ open. By definition of the topology on Y, this means that $f^{-1}(W)$ is open in Y. hence f is continuous.

10. *Is it true that for any topological space X and any identification space $\pi : X \to Y$, π maps open subsets of X to the open subsets of Y?*
No! Let U be an open subset of X. For $\pi(U)$ to be open in the identification space Y, we need $\pi^{-1}(\pi(U))$ to be open in X. Thus the identification map π maps open sets into open sets (i.e. is an open map) iff $\pi^{-1}(\pi(U))$ is open for any open subset U of X. But this latest version has nothing to do with the topology on Y, it is only a condition on the topology of X on the map π. Take for example $X = \mathbf{R}$ with the usual topology, $Y = \{0, 1\}$, $\pi(x) = 0$ if $x \in [0, 1]$ and $\pi(x) = 1$ otherwise. Then whatever the topology on Y, $\pi^{-1}(\pi(1/2, 3/4)) = [0, 1]$ which is not open.

11. *Let X and Y be two topological spaces and $\pi : X \to Y$ a continuous onto map. Show that if f maps the open (resp. closed) sets of X to open (resp. closed) sets of Y then Y is an identification space and that π is the identification map.*
Let τ be the topology on Y and σ be the topology on Y as the identification space. Since π is continuous and since σ is the largest topology that makes π continuous, $\tau \subseteq \sigma$, i.e. every open subset of Y for τ is an open subset for σ. Conversely, let V be an open subset of Y for σ. Then, by the definition of σ, $\pi^{-1}(V)$ is open. Since π maps the open sets of X onto the open subsets of Y for the topology given by τ, $\pi(\pi^{-1}(V))$ is open in the topology given by τ. But since π is onto, $\pi(\pi^{-1}(V)) = V$ and so V is open in τ.
The proof is similar for the second case.

12. *Assume X is compact and Y is Hausdorff. Let $\pi : X \to Y$ be a continuous onto map. Show that Y is an identification space and that π is the identification map.*
By the above question, we have to show that π maps the closed subsets of X onto the closed subsets of Y. Let C be a closed subset of X. Since X is compact, by number 2, C is compact. Since f is continuous, by number 4, $f(C)$ is compact in Y. Since Y is Hausdorff, by number 3, $f(C)$ is closed in Y. This proves the statement.

III. Topological Groups and Connected Components (40 pts.)
Recall that a topological space X is **connected** if it is not the union of the disjoint nonempty open subsets. Let X be a topological space.
13. *Let $A \subseteq X$ be a connected subspace of X. Show that $\overline{A}$ is connected.*
Assume $\overline{A} = U \sqcup V$ for two open subsets of $\overline{A}$. Then $A = (A \cap U) \sqcup (A \cap V)$. Since $A \cap U$ and $A \cap V$ are open in A, one of them must be empty, say $A \cap U$. Then A is in the closed subset U^{c} and so $\overline{A} \subseteq U^{c}$. Thus $\overline{A} \cap U = \emptyset$ amd $U = \emptyset$.

14. *Show that the relation "$x \equiv y$ iff x and y belong to a connected subspace of X" is an equivalence relation on X.*
Since $\{x\}$ is connected, the relation is reflexive. It is certainly symmetric. It remains to show that it is transitive. We will show that the union of two intersecting connected sets is connected, and this will prove the transitivity. Let A and B be two connected sets and assume that $x \in A \cap B$. Let $A \cup B = U \sqcup V$ for some open subsets U and V of $A \cup B$. Then

$$A = (A \cap U) \sqcup (A \cap V) \text{ and } B = (B \cap U) \sqcup (B \cap V)$$

Since $A \cap U$ and $A \cap V$ are open in A and $B \cap U$ and $B \cap V$ are open in B, by the connectivity of A and B one from each pair must be empty. Since $A \cap B \neq \varnothing$ and $U \cap V = \varnothing$, if $A \cap U = \varnothing$ then $B \cap V \neq \varnothing$ and so $B \cap U = \varnothing$, hence $U = \varnothing$. If $A \cap V = \varnothing$, similarly $V = \varnothing$.

15. *Show that each equivalence class for the above relation is a maximal connected subspace and is closed. Each equivalence class is called a* ***connected component*** *of X.*
One shows, exactly as above, that the union of connected sets which intersect nontrivially (even two by two) is a connected set. This shows that, given an element x, there is a maximal connected subset that contains it. The elements of the maximal connected subsets are clearly equivalent. That the maximal connected subsets are equivalence classes follows from the proof above.
By number 13, the connected components are closed.
It is wrong that the connected components are open as the following example shows: Let $X = \mathbf{Q}$. Then the connected subsets of X are only the singleton sets. Hence each connected component is a singleton and is not open.

A **topological group** G is both a Hausdorff topological space and a group such that the multiplication map $m : G \times G \to G$ and the inversion map $i : G \to G$ given by $m(x, y) = xy$ and $i(x) = x^{-1}$ are continuous.

16. *Show that in a topological group, the connected component of 1 is a closed normal subgroup.*
Let G be the topological group and C the connected component of 1. Let $x, y \in C$. Then $x \in C \cap xy^{-1}C$ and C and $xy^{-1}C$ are both connected. Thus $C \cup xy^{-1}C$ is connected and is equal to C. Thus $xy^{-1} \in xy^{-1}C \subseteq C$. This shows that C is a subgroup.
Considering the connected set xCx^{-1} (that contains 1) one sees as above that $xCx^{-1} = C$.
Hence C is a normal subgroup. It is closed by above. Therefore each coset of C is closed as well.