Projective Planes
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0. Let \( q \) be a prime power. Find the orders of \( \text{GL}_n(q), \text{PGL}_n(q), \text{SL}_n(q), \text{PSL}_n(q) \).

I. Let \( F \) be a division ring (a noncommutative field) and let \( K \) be a subdivision ring of \( F \). Assume that \( F \), as a left vector space over \( K \), is three dimensional. We may then consider the classical (Desarguesian) projective plane \( \pi = \pi(F/K) \) over the three dimensional \( K \)-vector space \( F \). Recall that the points and the lines of \( \pi \) are the left \( K \)-subspaces of dimension 1 and 2 of the three dimensional \( K \)-vector space \( F \) and that the incidence relation is given by the inclusion relation.

Let \( \text{GL}(F/K) \) be the group of left \( K \)-vector space automorphisms of \( F \). Clearly, the group \( \text{GL}(K/F) \) induces the group \( \text{PGL}(K/F) \) of automorphisms of \( \pi \).

I1. Show that the points of \( \pi \) are in one-to-one correspondence with the left-coset space \( F^*/K^* \).

For \( f \in F^* \), define the map \( R_f : F \to F \) by \( R_f(x) = xf \).

I2. Show that \( R_f \in \text{GL}(F/K) \).

I3. Show that \( \{R_f : f \in F^*\} \) is a subgroup of \( \text{GL}(K/F) \) isomorphic to \( F^* \).

Each \( R_f \), being in \( \text{GL}(F/K) \), induces an automorphism \( PR_f \in \text{PGL}(F/K) \) of \( \pi \).

I4. Find the kernel of the group homomorphism \( f \mapsto PR_f \) from \( F^* \) into \( \text{PGL}(F/K) \).

I5a. Give an example of \( F \) and \( K \) where \( K \) is a field.

I5b. Can \( K \) be a field without \( F \) being a field?

I5c. What is the kernel of the group homomorphism of question #3 in case \( F \) is a field.

I5d. Find explicitly the groups and the automorphisms of the questions above in case \( \pi \) is the projective plane of order 8.

II. Show that every involutary automorphism of a projective plane is either a homology, an elation or a Baer automorphism.

III. Let \( \pi \) be a projective plane. Let \( \alpha \) be an involutary \((A, a)\)-homology and \( \beta \) be an involutary \((B, b)\)-homology such that \( A \) is on \( b \) and \( B \) is on \( a \), with \( A \neq B \). Show that \( \alpha \beta \) is an involutary \((ab, AB)\)-homology.