Projective Planes

Ali Nesin
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0. Let q be a prime power. Find the orders of $\mathrm{GL}_n(q)$, $\mathrm{PGL}_n(q)$, $\mathrm{SL}_n(q)$, $\mathrm{PSL}_n(q)$.

I. Let F be a division ring (a noncomutative field) and let K be a subdivision ring of F. Assume that F, as a left vector space over K, is three dimensional. We may then consider the classical (Desarguesian) projective plane $\pi = \pi(F/K)$ over the three dimensional K-vector space F. Recall that the points and the lines of π are the left K-subspaces of dimension 1 and 2 of the three dimensional K-vector space F and that the incidence relation is given by the inclusion relation.

Let $\mathrm{GL}(F/K)$ be the group of left K-vector space automorphisms of F. Clearly, the group $\mathrm{GL}(K/F)$ induces the group $\mathrm{PGL}(K/F)$ of automorphisms of π.

I1. Show that the points of π are in one-to-one correspondance with the left-coset space F^*/K^*.

For $f \in F^*$, define the map $R_f : F \rightarrow F$ by $R_f(x) = xf$.

I2. Show that $R_f \in \mathrm{GL}(F/K)$.

I3. Show that $\{R_f : f \in F^*\}$ is a subgroup of $\mathrm{GL}(K/F)$ isomorphic to F^*.

Each R_f, being in $\mathrm{GL}(F/K)$, induces an automorphism $PR_f \in \mathrm{PGL}(F/K)$ of π.

I3. Find the kernel of the group homomorphism $f \mapsto PR_f$ from F^* into $\mathrm{PGL}(F/K)$.

I4. Show that the image of F^* under the above homomorphism acts regularly on the set of points of π.

I5a. Give an example of F and K where K is a field.

I5b. Can K be a field without F being a field?

I5c. What is the kernel of the group homomorphism of question #3 in case F is a field.

I5d. Find explicitely the groups and the automorphisms of the questions above in case π is the projective plane of order 8.

II. Show that every involutary automorphism of a projective plane is either a homology, an elation or a Baer automorphism.

III. Let π be a projective plane. Let α be an involutary (A, a)-homology and β be an involutary (B, b)-homology such that A is on b and B is on a, with $A \neq B$. Show that $\alpha\beta$ is an involutary (ab, AB)-homology.