Projective Planes

Final Exam
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π always denotes a projective plane.

A **correlation** of π is a bijection α from the points of π onto the lines of π and from the lines of π onto the points of π such that a point A is on a line ℓ iff the point ℓ^α is on the line A^α. Clearly a projective plane has a correlation if and only if it is self-dual.

1. Let $\varphi \in \mathrm{Aut}(\pi)$ and α, β be correlations of π. Show that

1a. $\alpha\beta \in \mathrm{Aut}(\pi)$.

1b. $\alpha\varphi$ and $\varphi\alpha$ are correlations of π.

1c. Show that $\alpha\mathrm{Aut}(\pi)$ is the set of all correlations of π.

A **polarity** is a correlation of order 2.

2. Show that if K is a field, the projective plane $\pi_2(K)$ has a "canonical" polarity.

If α is a polarity of π, then a point A (resp. a line ℓ) is called **absolute** if $A \in A^\alpha$ (resp. $\ell^\alpha \in \ell$).

3a. Give an example of a projective plane π with a polarity that has absolute points.

3b. Give an example of a projective plane π with a polarity that has no absolute points.

4. Let α be a polarity of π. Show that every absolute point of π is on a unique absolute line and vice versae.

5. Show that a finite projective plane π has a polarity iff the points and lines of π can be labelled in such a way that the incidence matrix of π with respect to this labelling is symmetric. Show that this labelling can be done in such a way that the number of ones on the diagonal (i.e. the trace of the incidence matrix) corresponds to the number of absolute points of the polarity.

6. Let F be a field. Let i be a new symbol. Define

$$F[i] = F \oplus Fi = \{x + yi : x, y \in F\}$$

artificially/formally. Define the addition and the multiplication by mimicing the complex numbers.

6a. Show that this may not always define a field structure.

6b. Show that the "norm" map N defined by $N(x + yi) = x^2 + y^2$ from the above algebraic structure into F is multiplicative.

6c. Let $\Delta = \{z \in F[i] : N(z) = 0\}$. If $F = \mathbf{F}_q$ (a finite field) compute the cardinality of Δ.

6d. Show that $F[i] \setminus \Delta$ is a group under the multiplication.

6e. What is the order of $F_q[i] \setminus \Delta$?

6f. Show that in a field F, the set $\{x^2 + y^2 : x, y \in F\} \setminus \{0\}$ form a multiplicative subgroup containing the set of nonzero squares F^{*2}.

6g. Show that in a finite field, every element is a sum of two squares, i.e. that the multiplicative map N is onto.

6h. Find the cardinality of the set $\{(x, y) \in \mathbf{F}_q{}^2 : x^2 + y^2 = 1\}$.

6i. Find the cardinality of the set $\{(x, y) \in \mathbf{F}_q{}^2 : x^2 + y^2 = -1\}$.

6j. Show that $\pi(\mathbf{F}_q)$ has a polarity (the canonical polarity would do) with at least $q + 1$ absolute points.

Discussion: It is not known whether a self-dual projective plane should necessarily have a polarity. It can be shown that a polarity on a finite projective plane of order n has at least $n + 1$ absolute points.