

# Projective Planes



Midterm  
5 Nisan 2000

**1.** Let  $\pi$  be a projective plane. Let  $\alpha$  be an involutory  $(A, a)$ -homology and  $\beta$  be an involutory  $(B, b)$ -homology such that  $A$  is on  $b$  and  $B$  is on  $a$ , with  $A \neq B$ . We will show that  $\alpha\beta$  is an involutory  $(ab, AB)$ -homology.

**1a.** Show that  $\alpha\beta$  fixes the points  $A, B$  and  $ab$ .

**1b.** Show that  $\alpha\beta$  is an involution (Hint:  $(\alpha\beta)^2$  fixes all the points of  $a \cup b$ ).

**1c.** Show that  $\alpha\beta$  cannot be an elation.

**1d.** Show that  $\text{Fix}(\alpha\beta) \cap \wp(a) = \{B, ab\}$ .

**1e.** Show that  $\alpha\beta$  cannot be a Baer collineation.

**1f.** Show that  $\alpha\beta$  is an involutory  $(ab, AB)$ -homology.

**2.** Let  $G$  be a group and let  $A$  and  $B$  be two subgroups of  $G$ . We define an incidence geometry  $\pi = \pi(G, A, B)$  as follows:

The points of  $\pi$  are the left cosets  $xA$  of  $A$  in  $G$ .

The lines of  $\pi$  are the left cosets  $yB$  of  $B$  in  $G$ .

A point  $xA$  is on a line  $yB$  iff  $xA \cap yB \neq \emptyset$ .

**2a.** Find the set  $\wp(B)$  of points of the line  $B$ . Find the set of points  $\wp(yB)$  of the line  $yB$ . Find the set of lines  $\mathcal{L}(xA)$  through the point  $xA$ .

**2b.** Show that the action of  $G$  on  $\wp(\pi)$  and on  $\mathcal{L}(\pi)$  by left multiplication gives rise to automorphisms of the incidence geometry  $\pi$  and that this action is transitive on  $\wp(\pi)$ , and also on  $\mathcal{L}(\pi)$ .

**2c.** When is the action of  $G$  on the set  $\wp(\pi)$  of points of  $\pi$  faithful? The same question for the set  $\mathcal{L}(\pi)$  of lines of  $\pi$ .

**2d.** Show that  $G$  acts transitively on the set  $\mathfrak{F}(\pi) = \{(P, \ell) : P \in \ell\}$  of flags of  $\pi$ . Show that this action is regular iff  $A \cap B = 1$ .

**2e.** Show that the stabilizer of the point  $A$  of  $\pi$  is  $A$ .

**2f.** Show that any two distinct lines of  $\pi$  intersect if and only if  $G = BAB$ .

**2g.** Show that a line passes through any two distinct points if and only if  $G = ABA$ .

**2h.** Find the necessary and sufficient conditions on  $G, A$  and  $B$  so that  $\pi$  is a projective plane.