Projective Planes

Midterm
5 Nisan 2000

1. Let π be a projective plane. Let α be an involutary (A, a)-homology and β be an involutary (B, b)-homology such that A is on b and B is on a, with $A \neq B$. We will show that $\alpha\beta$ is an involutary (ab, AB)-homology.

1a. Show that $\alpha\beta$ fixes the points A, B and ab.

1b. Show that $\alpha\beta$ is an involution (Hint: $(\alpha\beta)^2$ fixes all the points of $a \cup b$).

1c. Show that $\alpha\beta$ cannot be an elation.

1d. Show that $\text{Fix}(\alpha\beta) \cap \wp(a) = \{B, ab\}$.

1e. Show that $\alpha\beta$ cannot be a Baer collineation.

1f. Show that $\alpha\beta$ is an involutary (ab, AB)-homology.

2. Let G be a group and let A and B be two subgroups of G. We define an incidence geometry $\pi = \pi(G, A, B)$ as follows:

The points of π are the left cosets xA of A in G.

The lines of π are the left cosets yB of B in G.

A point xA is on a line yB iff $xA \cap yB \neq \emptyset$.

2a. Find the set $\wp(B)$ of points of the line B. Find the set of points $\wp(yB)$ of the line yB. Find the set of lines $\mathcal{L}(xA)$ through the point xA.

2b. Show that the action of G on $\wp(\pi)$ and on $\mathcal{L}(\pi)$ by left multiplication gives rise to automorphisms of the incidence geometry π and that this action is transitive on $\wp(\pi)$, and also on $\mathcal{L}(\pi)$.

2c. When is the action of G on the set $\wp(\pi)$ of points of π faithful? The same question for the set $\mathcal{L}(\pi)$ of lines of π.

2d. Show that G acts transitively on the set $\Im(\pi) = \{(P, \ell) : P \in \ell\}$ of flags of π. Show that this action is regular iff $A \cap B = 1$.

2e. Show that the stabilizer of the point A of π is A.

2f. Show that any two distinct lines of π intersect if and only if $G = BAB$.

2g. Show that a line passes through any two distinct points if and only if $G = ABA$.

2h. Find the necessary and sufficient conditions on G, A and B so that π is a projective plane.