Math 281

Midterm
Kasım 1998
Ali Nesin

1. Does the ring $\mathbb{R}[X]/<X^2 - 1>$ have zerodivisors? Find them. Find its units.

2a. Show that a finite field has p^n elements for some prime p and some $n > 0$. (Hint: A field is a vector space over its prime field).

2b. Show that in a finite field with p^n elements, $x^{p^n} = x$ for all x.

2c. Show that in a finite field with p^n elements where p is odd, there are $\frac{p^n - 1}{2}$ many elements with a square root.

2d. Let F be a finite field of characteristic p (a prime). Show that $x \to x^p$ is an automorphism of F.

3. Show that the polynomial $f(x) = x^4 + x + 1$ is irreducible over $\mathbf{F}_2$.

3a. Find a field over which f is reducible.

3b. Show that if f is reducible over a field F, then one of the following polynomials has a root in F: $x^4 + x + 1$, $x^2 + 1$, $x^2 - 2$, $x^2 + 2$.

3c. Find the elements of $\mathbf{F}_2[X]/<X^4 + X + 1>$. Show its multiplication table.

4. Let $V = \{(x - y + z + t, x + y + z + t, z + t, x + z + t): x, y, z, t \in \mathbb{R}\}$. V is clearly a vector space over the field $\mathbb{R}$. What is its dimension? Find a basis of V. Complete this basis to a basis of $\mathbb{R}^4$.