Math 211

Algebra
First Midterm
November 2000
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G always denotes a group.

1a. Let R be a (not necessarily commutative) ring with 1. Show that the set R^* of invertible elements of R form a group under multiplication. (2 pts.)

1b. Find the set $(\mathbb{Z}/24\mathbb{Z})^*$ and draw its multiplication table. (4 + 4 pts.)

1d. Find $(\mathbb{R}[X]/\langle X^3\rangle)^*$ explicitely and show that it is isomorphic to the following group of matrices (8 + 8 pts.):

$$\left\{\begin{pmatrix}1 & a & b\\ 0 & 1 & a\\ 0 & 0 & 1\end{pmatrix} : a, b \in \mathbb{R}\right\}$$

2. Let R be a commutative ring and $I \lhd R$ an ideal. Define

$$\sqrt{I} = \{r \in R : r^n \in I \text{ for some integer } n\}$$

Show that $\sqrt{I}$ is an ideal of R. (5 pts.)

3. Let H be a subgroup of index 2 of G (i.e. $|G/H| = 2$). Show that $H \lhd G$. (6 pts.)

4. Let H and K be two subgroups of G. Show that for x and y in G, $xH \cap yK$ either is empty or a coset of $H \cap K$. (10 pts.)

5a. Let A and B be two groups. Assume B is abelian. Let End(A, B) be the set of group endomorphisms from A into B. For $f, g \in$ Hom(A, B) define $f + g$ by the rule

$$(f.g)(a) = f(a)g(a) \qquad (a \in A).$$

Show that the set of group endomorphisms Hom(A, B) from A into B form an abelian group under the addition of functions. (4 pts.)

5b. Let n and m be two integers > 0. Show that $\text{Hom}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}) \approx m'\mathbb{Z}/m\mathbb{Z} \approx \mathbb{Z}/d\mathbb{Z}$ where $m' = m/d$ and $d = \gcd(m, n)$. (8 + 8 pts.)

Hint: $\mathbb{Z}/n\mathbb{Z}$ is cyclic and generated by 1, this means that any endomorphism $\varphi \in \text{End}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z})$ is determined by its image $\varphi(1) \in \mathbb{Z}/m\mathbb{Z}$.

6. Let G be a group. Two elements g and h of G are said to be **conjugate** if $g^x := x^{-1}gx = h$ for some $x \in G$. For $g \in G$, we let the **centralizer** of g to be:

$$C_G(g) = \{x \in G : gx = xg\}.$$

6a. Show that to be conjugate is an equivalence relation. (4 pts.)

The class of g is called the **conjugacy class** of g and is denoted by g^G. Thus

$$g^G = \{g^x : x \in G\}.$$

6b. Show that $C_G(g) \leq G$. (3 pts.)

6c. Let $G/C_G(g)$ denote the the right coset space of $C_G(g)$ in G. Show that the map

$C_G(g)x \mapsto g^x$ is a well-defined bijection between $G/C_G(g)$ and g^G. (6 pts.)

6d. Sym(n) denotes the set of all bijections of the set $\{1, 2, \ldots, n\}$ and is called the **symmetric group** on n. An element of Sym(n) is called a **transposition** if it interchanges two distinct elements i and j of the set $\{1, 2, \ldots, n\}$ and fixes all the rest. Such a transposition is denoted (i, j) or (j, i). Show that the set of transpositions of Sym(n) is one conjugacy class. (8 pts.)

6e. Compute the order of the centralizer $C_{\mathrm{Sym}(n)}(1\ 2)$ of the element $(1, 2)$. (4 pts.)

6f. Show that $C_{\mathrm{Sym}(n)}(1\ 2) \approx \mathbb{Z}/2\mathbb{Z} \times \mathrm{Sym}(n-2)$. (8 pts.)