Math 411. Field Theory  
Final Exam, Jan. 2009  
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First Part
1. Let $K \leq L \leq K[\alpha]$ where $\alpha$ is algebraic over $K$.
   1a. Show that $L = K[\beta]$ for some $\beta$.
   1b. Let $x^n + a_{n-1}x^{n-1} + \ldots + a_0$ be the minimal monic polynomial of $\alpha$ over $L$. Show that $L = K[a_0, \ldots, a_{n-1}]$.
2. Show that for any field $K$ of characteristic $p > 0$, $[K(x) : K(x^p)] = p$. (Here $K(x)$ is the transcendental extension of $K$.)
3. Let $K$ be a field of characteristic $p > 0$. Let $M = K(x, y)$ and $L = K(x^p, y^p)$. Show that $[M : L] = p^2$ and that $[L(\alpha) : L] = p$ for any $\alpha \in M \setminus L$.
4. Let $p \in \mathbb{Z}$ be a prime. Let $K$ be the splitting field of $x^5 - p^2x + p$ over $\mathbb{Q}$. Find $\text{Gal}(K/\mathbb{Q})$.
5. Let $K$ be the splitting field of $x^5 + x^2 - x + 1$ over $\mathbb{Q}$. Find $\text{Gal}(K/\mathbb{Q})$.
6. Let $K$ be a field. Let $\Sigma = \{f : K \to K : f$ is multiplicative$\}$. Show that $\Sigma$ is a linearly independent set in the $K$-vector space of functions from $K$ into $K$.

Second Part. Below, $K \leq F$ symbolizes a field extension of characteristic $p > 0$.
7. Find the kernel of the group endomorphism $x \mapsto x^p - x$ of $F^\times$.
8. Let $\alpha \in F$ and $a = \alpha^p - \alpha$. Show that all the roots of $x^p - x - a$ are in $F$ and find them.
9. Let $a \in K$. Show that if the polynomial $x^p - x - a$ has no roots in $K$ then it is irreducible over $K$.
10. Let $a \in K$ be such that the polynomial $f(x) = x^p - x - a$ has no roots in $K$. Assume $F = K(\alpha)$ where $\alpha$ is a root of $f$.
    a) Show that $F$ is a splitting field of $f$ over $K$.
    b) Show that $F/K$ is a Galois extension.
    c) Show that there is an automorphism $\sigma$ of $F$ over $K$ such that $\sigma(\alpha) = \alpha + 1$.
    d) Show that $\text{Gal}(F/K) = \langle \sigma \rangle$ and has order $p$.
11. Let $F/K$ be Galois. Let $G = \text{Gal}(F/K)$. For $\alpha \in F$ define define the trace of $\alpha$ over $K$ as follows:
    \[ T(\alpha) = \sum_{\sigma \in G} \sigma(\alpha). \]
    Then $T$ is clearly an endomorphism of $F^\times$.
    a) What is $T(\alpha)$ for $\alpha \in K$?
    b) Show that $T(F) \subseteq K$.
    c) When can we have $T(K) = 0$?
    d) Show that $T \neq 0$.
    e) Show that $T(\alpha - \sigma(\alpha)) = 0$ for all $\sigma \in G$ and $\alpha \in F$.
12. Let $F/K$ be a cyclic Galois extension of degree $p$. Let $\text{Gal}(F/K) = \langle \sigma \rangle$. Let $\alpha \in F$ be such that $\sigma(\alpha) = \alpha + 1$. Let $a = \alpha^p - \alpha$.
    a) Show that $a \in K$.
    b) Show that $x^p - x - a$ is the monic irreducible polynomial of $\alpha$ over $K$.
    c) Show that $F = K(\alpha)$.
13. Let $F/K$ be a cyclic Galois extension. Let $G = \text{Gal}(F/K) = \langle \sigma \rangle$. Assume $|G| = n$. Let $\beta \in F$ be such that $T(\beta) = 0$. Set $\gamma_i = \beta + \sigma(\beta) + \ldots + \sigma^i(\beta)$. Let $\gamma \in F$ with $T(\gamma) \neq 0$ (show that such an element exists). Let
\[ \alpha = T(\gamma)^{-1}(\gamma_0 \gamma + \gamma_1 \sigma(\gamma) + \ldots + \gamma_{n-1} \sigma^{n-1}(\gamma)). \]

Check that \( \beta = \alpha - \sigma(\alpha) \). Conclude that for an element \( \beta \in F \), \( T(\beta) = 0 \) iff \( \beta = \alpha - \sigma(\alpha) \) for some \( \alpha \in F \).

14. Let \( F/K \) be an extension of degree \( p \). Show that \( F/K \) is a cyclic Galois extension iff there exists an \( \alpha \in F \) whose minimal polynomial is of the form \( x^p - x - a \) for some \( a \in K \) and such that \( F = K(\alpha) \).