Math 411. Field Theory
Final Exam, Jan. 2009
Ali Nesin

First Part

1. Let $K \leq L \leq K[\alpha]$ where α is algebraic over K.

1a. Show that $L = K[\beta]$ for some β.

1b. Let $x^n + a_{n-1}x^{n-1} + ... + a_0$ be the minimal monic polinomial of α over L. Show that $L = K[a_0, ..., a_{n-1}]$.

2. Show that for any field K of characteristic $p > 0$, $[K(x) : K(x^p)] = p$. (Here $K(x)$ is the transcendantal extension of K.)

3. Let K be a field of characteristic $p > 0$. Let $M = K(x, y)$ and $L = K(x^p, y^p)$. Show that $[M : L] = p^2$ and that $[L(\alpha) : L] = p$ for any $\alpha \in M \setminus L$.

4. Let $p \in \mathbb{Z}$ be a prime. Let K be the splitting field of $x^5 - p^2x + p$ over $\mathbb{Q}$. Find $\mathrm{Gal}(K/\mathbb{Q})$.

5. Let K be the splitting field of $x^5 + x^2 - x + 1$ over $\mathbb{Q}$. Find $\mathrm{Gal}(K/\mathbb{Q})$.

6. Let K be a field. Let $\Sigma = \{f : K \to K : f \text{ is multiplicative}\}$. Show that Σ is a linearly independent set in the K-vector space of functions from K into K.

Second Part. Below, $K \leq F$ symbolizes a field extension of characteristic $p > 0$.

7. Find the kernel of the group endomorphism $x \mapsto x^p - x$ of F^+.

8. Let $\alpha \in F$ and $a = \alpha^p - \alpha$. Show that all the roots of $x^p - x - a$ are in F and find them.

9. Let $a \in K$. Show that if the polinomial $x^p - x - a$ has no roots in K then it is irreducible over K.

10. Let $a \in K$ be such that the polinomial $f(x) = x^p - x - a$ has no roots in K. Assume $F = K(\alpha)$ where α is a root of f.

a) Show that F is a splitting field of f over K.

b) Show that F/K is a Galois extension.

c) Show that there is an automorphism σ of F over K such that $\sigma(\alpha) = \alpha + 1$.

d) Show that $\mathrm{Gal}(F/K) = \langle\sigma\rangle$ and has order p.

11. Let F/K be Galois. Let $G = \mathrm{Gal}(F/K)$. For $\alpha \in F$ define define the ***trace*** of α over K as follows:

$$T(\alpha) = \sum_{\sigma \in G} \sigma(\alpha) .$$

Then T is clearly an endomorphism of F^+.

a) What is $T(\alpha)$ for $\alpha \in K$?

b) Show that $T(F) \subseteq K$.

c) When can we have $T(K) = 0$?

d) Show that $T \neq 0$.

e) Show that $T(\alpha - \sigma(\alpha)) = 0$ for all $\sigma \in G$ and $\alpha \in F$.

12. Let F/K be a cyclic Galois extension of degree p. Let $\mathrm{Gal}(F/K) = \langle\sigma\rangle$. Let $\alpha \in F$ be such that $\sigma(\alpha) = \alpha + 1$. Let $a = \alpha^p - \alpha$.

a) Show that $a \in K$.

b) Show that $x^p - x - a$ is the monic irreducible polynomial of α over K.

c) Show that $F = K(\alpha)$.

13. Let F/K be a cyclic Galois extension. Let $G = \mathrm{Gal}(F/K) = \langle\sigma\rangle$. Assume $|G| = n$. Let $\beta \in F$ be such that $T(\beta) = 0$. Set $\gamma_i = \beta + \sigma(\beta) + ... + \sigma^i(\beta)$. Let $\gamma \in F$ with $T(\gamma) \neq 0$ (show that such an element exists). Let

$$\alpha = T(\gamma)^{-1}(\gamma_0\gamma + \gamma_1\sigma(\gamma) + \ldots + \gamma_{n-1}\sigma^{n-1}(\gamma)).$$

Check that $\beta = \alpha - \sigma(\alpha)$. Conclude that for an element $\beta \in F$, $T(\beta) = 0$ iff $\beta = \alpha - \sigma(\alpha)$ for some $\alpha \in F$.

14. Let F/K be an extension of degree p. Show that F/K is a cyclic Galois extension iff there exists an $\alpha \in F$ whose minimal polynomial is of the form $x^p - x - a$ for some $a \in K$ and such that $F = K(\alpha)$.