Math 231: Linear Algebra I Final
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1. Let V be a finite dimensional vector space. Let $A, B \in \text{End } V$ be such that $AB = \text{Id}_V$. Show that $BA = \text{Id}_V$. Show that this is false if V is infinite dimensional. (5 pts.)
Proof: Clearly B is one to one. Hence it is also onto. Since V is finite dimensional, B is then invertible. Let C be the inverse of B. Since $AB = \text{Id}_V = CB$, we have $(A - C)B = 0$. Since B is invertible, it follows that $A = C = B^{-1}$. Therefore $BA = BB^{-1} = \text{Id}_V$. If V is infinite dimensional this is false. Indeed let $V = K[X]$ and A and B be defined by $A(f) = (f - f(0))/X$ and $B(f) = Xf$.

2. Let V be a vector space of dimension $n < \infty$. A sequence $V_0 < ... < V_k$ of subspaces of V is called a ***flag*** and k is called the ***length*** of that flag.
2a. What is the maximal possible length of a flag? Construct a flag of maximal length (3 pts.)
Answer: Since $0 \leq \dim V_0 < ... < \dim V_k \leq \dim V = n$, we must have $k \leq n$. If $v_1, ..., v_n$ is a basis of V, for each $i = 1, ..., n$ set $V_i = \langle v_1, ..., v_i \rangle$. Then $(V_i)_i$ is a flag of length n.
2b. Show that $\text{GL}(V)$ acts naturally on the set of flags of length k. (2 pts.)
Proof: Any flag is of course sent to a flag by an element of $\text{GL}(V)$.

3. Let V be a vector space of dimension $n < \infty$. Let $v_1, ..., v_n$ be a basis of V. For $\alpha \in \text{Sym}(n)$, define $f(\alpha) = f_\alpha \in \text{GL}(V)$ in such a way that $f_\alpha(v_i) = v_{\alpha(i)}$ for all $i = 1, ..., n$.
3a. Why such an $f(\alpha)$ must exist? (2 pts.)
Answer: Because $v_1, ..., v_n$ form a basis of V.
3b. Show that f is a group homomorphism from $\text{Sym}(n)$ into $\text{GL}(V)$. (3 pts.)
Proof: Let $\alpha, \beta \in \text{Sym } n$. Since $f_{\alpha\beta}(v_i) = v_{\alpha\beta(i)} = f_\alpha(v_{\beta(i)}) = f_\alpha f_\beta(v_i)$ for all i, we have $f_{\alpha\beta}(v) = f_\alpha f_\beta(v)$ for all $v \in V$. Hence $f_{\alpha\beta} = f_\alpha f_\beta$.

4. Let V be a vector space of dimension n and W be a subspace of dimension k. It must be clear that $A := \{g \in \text{End}(V) : gW \leq W\}$ is a vector space. Find its dimension. (6 pts.)
Proof: Choose a basis $w_1, ..., w_k$ of W and complete it to basis of V by adding $v_{k+1}, ..., v_n$. An element of A is determined by what it does to this basis. Each of the w_i's must go to a linear combination of $w_1, ..., w_k$ and this adds k^2 to the basis. Each of the v_j's must go to a linear combination of $w_1, ..., w_k, v_{k+1}, ..., v_n$.and this adds $(n - k)n$ to the basis. Thus $\dim_K A = k^2 + (n - k)n = n^2 - nk + k^2$.

5. Let V be a vector space of dimension $n < \infty$. Let $v_1, ..., v_n$ be a basis of V. For each $i = 1, ..., n$ let $V_i = \langle v_1, ..., v_i \rangle$. Let $A = \{g \in \text{End}_K(V) : gV_i \leq V_i \text{ for all } i = 1, .., n\}$. It must be clear that A is an algebra over K. Find $\dim_K(A)$. (4 pts.)
Proof: For each i, $g(v_i)$ must be a linear combination of $v_1, ..., v_i$. Hence $\dim A = 1 + 2 + ... + n = n(n+1)/2$.

6. Let V be a vector space of dimension $n < \infty$ over a field K. For a positive integer k and a map f from $V^k = V \times ... \times V$ into K is called *k-multilinear* if f is linear in each coordinate. Let $E_k(V)$ be the set of k-multilinear maps of V. $E_k(V)$ is a vector space.
6a. What is $\dim E_k(V)$? (6 pts.)
Proof: Let $v_1, ..., v_n$ be a basis of V. An element of $E_k(V)$ is determined by what it does to the set $\{(v_{i_1}, ..., v_{i_k}) : i_1, ..., i_k \in \{1, ..., n\}\}$. Thus $\dim E_k(V) = n^k$.

6b. A k-multilinear map f from $V \times ... \times V$ into K is called *k-alternating* if for any w_1, ..., $w_k \in V$, $f(w_1, ..., w_k) = 0$ whenever $w_i = w_j$ for two distinct i and j. Let $A_k(V)$ be the set of k-alternating maps of V**.** Then $A_k(V)$ is a vector space. Show that for any w_1, ..., $w_k \in V$, $f \in A_k(V)$ and any $\sigma \in \text{Sym } k$,

$$f(w_{\sigma(1)}, ..., w_{\sigma(k)}) = \text{sg}(\sigma) f(w_1, ..., w_k).$$

(Here sg(σ) is the ***signature*** of σ, i.e. is 1 or -1, depending on whether $\sigma \in \text{Alt } n$ or not, in other words, $\text{sg}(\sigma) = (-1)^{\ell}$ where ℓ is the number of transpositions whose product is σ). (6 pts.)

6c. What is dim $A_k(V)$? (10 pts.)

Answer and Proof: Let v_1, ..., v_n be a basis of V. Using 8b, it is easily seen that an element of $A_k(V)$ is determined by what it does to the set

$$\{(v_{i_1}, ..., v_{i_k}) : 1 \le i_1 < ... < i_k \le n\}$$

which is in bijection with the set of k-subsets of $\{1, ..., n\}$. Thus dim $A_k(V) = \binom{n}{k}$. It follows that dim $A_k(V) = 0$ if $k > n$ and dim $A_n(V) = 1$.

7. Let V be a vector space and W be a subspace of V.

7a. Let $(w_i)_{i\in I} \cup (v_j)_{j\in J}$ be a basis of V such that $(w_i)_{i\in I}$ is a basis of W. Show that $(\overline{v_j})_{j\in J}$ is a basis of V/W. (3 pts.)

Proof: First the linear independence: Assume $\sum_j \alpha_j \overline{v_j} = \overline{0}$. Then $\overline{\sum_j \alpha_j v_j} = \overline{0}$ and so $\sum_j \alpha_j v_j \in W$. Since $(w_i)_{i\in I} \cup (v_j)_{j\in J}$ be a basis of V, this shows that $\alpha_j = 0$ for all j. That $(\overline{v_j})_{j\in J}$ generates V/W is much easier.

7b. Let $(w_i)_{i\in I}$ be a basis of W. Let $(v_j)_{j\in J}$ be such that $(\overline{v_j})_{j\in J}$ is a basis of V/W. Show that $(w_i)_{i\in I} \cup (v_j)_{j\in J}$ be a basis of V. (3 pts.)

Proof: First the linear independence: Assume $\sum_i \alpha_i w_i + \sum_j \beta_j v_j = 0$. Taking modulo W we get $\sum_j \beta_j \overline{v_j} = \overline{0}$. Since $(\overline{v_j})_{j\in J}$ is a basis of V/W, we get $\beta_j = 0$ all j. Hence $\sum_i \alpha_i w_i = \sum_i \alpha_i w_i + \sum_j \beta_j v_j = 0$ and since $(w_i)_{i\in I}$ be a basis of W we get $\alpha_i = 0$ all i.

Generating: Let $v \in V$. Then $\overline{v} = \sum_j \alpha_j \overline{v_j}$ for some α_i. Then $v - \sum_j \alpha_j v_j \in W$ and the rest is easy.

8. Let V be a vector space and $W \le V$. Let $g \in GL(V)$.

8a. Show that g induces naturally an isomorphism from $V/g^{-1}(W)$ onto V/W. (4 pts.)

Proof: Define $\overline{g} : V/g^{-1}(W) \to V/W$ via $\overline{g}(\overline{\overline{v}}) = \overline{g(v)}$. We must show that this is well-defined: For $v_1, v_2 \in V$, we have

$$\overline{\overline{v_1}} = \overline{\overline{v_2}} \Leftrightarrow v_1 - v_2 \in g^{-1}(W) \Leftrightarrow g(v_1 - v_2) \in W \Leftrightarrow g(v_1) - g(v_2) \in W$$
$$\Leftrightarrow \overline{g(v_1)} = \overline{g(v_2)}.$$

This shows that $\overline{g}$ is both well-defined and one-to-one. Since g is onto, it is clear that $\overline{g}$ is onto as well.

8b. Show that $H := \{g \in GL(V) : g(W) = W\}$ is a subgroup GL(V) and that there is a natural grup homomorphism φ from H into GL(V/W). (4 pts.)

Proof: Clearly H is a subgroup. From part a, the map φ defined by $\varphi(g) = \overline{g}$ from H into GL(V/W) is well-defined. To show it is a homomorphism of groups, we compute: $\varphi(ab)(\overline{v}) = \overline{(ab)(v)} = \overline{a(b(v))} = \overline{a}(\overline{b(v)}) = \overline{a}(\overline{b}(\overline{v})) = (\overline{a} \circ \overline{b})(\overline{v}) = (\varphi(a) \circ \varphi(b))(\overline{v})$ for all $\overline{v} \in V/W$. Thus $\varphi(ab) = \varphi(a)\circ\varphi(b)$.

8c. Find the kernel of the above homomorphism φ. (2 pts.)

Proof: $g \in \text{Ker}\,\varphi \Leftrightarrow \varphi(g) = \text{Id}_{V/W} \Leftrightarrow \overline{g} = \text{Id}_{V/W} \Leftrightarrow \overline{g}(\overline{v}) = \overline{v}$ for all $v \in V \Leftrightarrow \overline{g(v)} = \overline{v}$ for all $v \in V \Leftrightarrow g(v) - v \in W$ for all $v \in V \Leftrightarrow g(v) \in v + W$ for all $v \in V$. Hence,

$$\text{Ker}\,\varphi = \{g \in \text{GL}(V) : g(v) \in v + W \text{ for all } v \in V\}.$$

(Note that such a g is necessarily in H, just take $v \in W$.

8d. Let $\psi \in \text{GL}(V/W)$. Let $(w_i)_{i\in I} \cup (v_j)_{j\in J}$ be a basis of V such that $(w_i)_{i\in I}$ is a basis of W. For $j \in J$, let $u_j \in V$ be such that $\psi(\overline{v_j}) = \overline{u_j}$. Show that $(w_i)_{i\in I} \cup (u_j)_{j\in J}$ is a basis of V. (7 pts.)

Proof: We first show the linear independence: Suppose $\sum_{i\in I}\alpha_i w_i + \sum_{j\in J}\beta_j u_j = 0$. Computing modulo W, we get $\psi(\sum_{j\in J}\beta_j \overline{v_j}) = \sum_{j\in J}\beta_j\psi(\overline{v_j}) = \sum_{j\in J}\beta_j \overline{u_j} = \overline{0}$. Since ψ is one-to-one, this shows that $\sum_{j\in J}\beta_j\overline{v_j} = \overline{0}$. Hence $\sum_{j\in J}\beta_j v_j \in W$. By the choice of $(v_j)_j$, this implies that $\beta_j = 0$ for all j. Hence $\sum_{i\in I}\alpha_i w_i = \sum_{i\in I}\alpha_i w_i + \sum_{j\in J}\beta_j u_j = 0$ and so by the choice of $(w_i)_i$, we get $\alpha_i = 0$ for all i. This shows the linear independence.

Now we show that the set generates V. Let $v \in V$. Since ψ is onto, there is a $\overline{u} \in V/W$ such that $\psi(\overline{u}) = \overline{v}$. Since $(\overline{v_j})_j$ generates V/W there are finitely many scalars α_j such that $\overline{u} = \sum_j \alpha_j \overline{v_j}$. Then

$$\overline{v} = \psi(\overline{u}) = \psi(\textstyle\sum_j \alpha_j \overline{v_j}.) = \sum_j \alpha_j \psi(\overline{v_j}) = \sum_j \alpha_j \overline{u_j} = \overline{\sum_j \alpha_j u_j}.$$

Therefore $v - \sum_j \alpha_j u_j \in W$ and we are done.

8e. Show that φ is onto. (15 pts.)

Proof: Let $\psi \in \text{GL}(V/W)$. Let $(w_i)_{i\in I} \cup (v_j)_{j\in J}$ be a basis of V such that $(w_i)_{i\in I}$ is a basis of W. For $j \in J$, let $u_j \in V$ be such that $\psi(\overline{v_j}) = \overline{u_j}$. We know by the previous question that $(w_i)_{i\in I} \cup (u_j)_{j\in J}$ is a basis of V. Define $g \in \text{End}\,V$ by $g(w_i) = w_i$ and $g(v_j) = u_j$. Since g sends a basis to a basis, it is clear that $g \in \text{GL}(V)$. Also $g|_W = \text{Id}_W$ and $\varphi(g) = \psi$.

8f. Find a subgroup of GL(V) which is naturally isomorphic to GL(V/W). (15 pts.)

Proof: Above, we showed in fact that for all $\psi \in \text{GL}(V/W)$ there is a $g \in G$ such that $\varphi(g) = \psi$ and $g|_W = \text{Id}_W$. Let $K = \{g \in \text{GL}(V) : g|_W = \text{Id}_W\}$. It is clear that $K \leq \text{GL}(V)$. The restriction of φ to K is onto by 4e. But this restriction of φ is not one-to-one in general. Let $U \leq V$ be a complement of W in V. Let

$$L = \{g \in \text{GL}(V) : g|_W = \text{Id}_W \text{ and } g(U) = U\}.$$

Then the restriction of φ to L is an isomorphism as can be shown easily.