Linear Algebra MT
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1. Find vector spaces U and V and a one-to-one linear map $\varphi : U \to V$ which is not onto.
2. Find vector spaces U and V and a surjective linear map $\varphi : U \to V$ which is not one-to-one.
3. Let V be a vector space. Let U and W be subspaces of V. Show that $(U + W)/W \approx U/(U \cap W)$.
4. Let $V = \mathbb{R}^3$. **4a.** Show that the vectors $v_1 = (1, 1, 0)$, $v_2 = (1, 2, -1)$, $v_3 = (0, -1, 2)$ are linearly independent. **4b.** For any $v \in V$, find $a(v), b(v), c(v) \in \mathbb{R}$ such that $v = a(v)v_1 + b(v)v_2 + c(v)v_3$. **4c.** Let $\varphi : V \to V$ be defined by $\varphi(x, y, z) = (x - y, y + 2z, x + y)$. Write the matrix of φ with respect to the bases v_1, v_2, v_3.
5. Let V be a vector space of dimension n over a field F. Let $W \le V$. **5a.** Show that $\{\varphi \in \text{End}_K(V) : \varphi(W) \le W\}$ is a subalgebra, say P_W of $\text{End}_K(V)$. **5b.** Find a relationship between $\dim P_W$, $\dim W$ and $\dim V$.
6. Let V be a vector space over a field K. Let V^* be the vector space of linear maps from V into K. Let $(v_i)_{i\in I}$ be a basis of V.
   6a. For $i \in I$ show that there is a unique $v_i^* \in V^*$ such that $v_i^*(v_j) = \delta_{i,j}v_j$ for all $j \in I$. Show that $(v_i^*)_{i\in I}$ is a linearly independet family of elements of V^*.
   6b. Show that if V is finite dimensional then so is V^* and that $\dim V = \dim V^*$.
   6c. Show that the set $(v_i^*)_{i\in I}$ of 4a is not a basis of V^* if $\dim V^* = \infty$.
   6d. For $A \subseteq I$ define $v_A^*(v_i) = \delta_A(i)v_i$ where δ_J is the characteristic function of J, i.e. $\delta_A(i)$ is equal to 1 or 0 depending on whether $i \in A$ or not. Assume I is infinite. Let $\wp$ be a set of infinite subsets of I such that for any two distinct $A, B \in \wp$, $A \cap B$ is finite. Show that $(v_A^*)_{A\in \wp}$ is a linearly independent set of vectors.
   6e. Show that $\mathbb{N}$ has uncountably many (in fact $|\mathbb{R}|$ many) subsets such that the intersection of any two distinct ones is finite.
   6f. Deduce from 4d and 4e that if $\dim V = \infty$ then $\dim V^* \ge |\mathbb{R}|$.
7. Let T be a set. Let $\ell^\infty(T) = \{f : T \to \mathbb{R} : f \text{ is bounded}\}$. Show that $\ell^\infty(T)$ is a vector space. For $f \in \ell^\infty(T)$, define $\|f\| = \sup\{|f(t)| : t \in T\}$. Show that for all $f, g \in \ell^\infty(T)$, and $\lambda \in \mathbb{R}$, we have,
   7a. $\|f\| \ge 0$ for all f.
   7b. $\|f\| = 0$ iff $f = 0$.
   7c. $\|\lambda f\| = |\lambda|\,\|f\|$.
   7d. $\|f + g\| \le \|f\| + \|g\|$.

   A vector space V over $\mathbb{R}$ together with a map $\|\ \| : V \to \mathbb{R}$ that satisfies 4a, 4b, 4c, 4d is called a ***normed vector space***. Thus $\ell^\infty(T)$ is a normed vector space together with the map $\|\ \|$ defined above.
8. Let $(V, \|\ \|)$ be a normed vector space. For $v, w \in V$ define $d(v, w) = \|v - w\|$. Show that (V, d) is a metric space.
9. Show that the normed vector space $\ell^\infty(T)$ defined in #4 is complete with respect to the norm defined in #5.