Linear Algebra HW
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Ali Nesin

1. Show that $\mathbb{Z}[X]/\langle 2X - 1\rangle$ is not a finitely generated $\mathbb{Z}$-module.
2. Let $f \in \mathbb{Z}[X]$ be a nonzero polynomial. When is $\mathbb{Z}[X]/\langle f\rangle$ a finitely generated $\mathbb{Z}$-module?
3. Let K be a field and $f \in K[X]$ be a polynomial of degree n. Show that $K[X]/\langle f\rangle$ is a K-vector space of dimension n.
4. Let V be a vector space of dimension 2. Let $\{v_1, v_2\}$ be a basis of V. When is $\{v_1 - v_2, v_1 + v_2\}$ a basis of V?
5. Let V be a vector space of dimension n. Let $v_1, v_2, ..., v_n$ be a basis of V. Show that
$$v_1, v_1 + v_2, v_1 + v_2 + v_3, ..., v_1 + v_2 + ... + v_n$$
is a basis of V.
6. Let V be a vector space of dimension n. Let W_1 and W_2 be two subpaces such that $W_1 \cap W_2 = 0$. What can you say about the dimensions of W_1 and W_2?
7. Let K be a finite field with q elements and V a K-vector space of dimension n.
    - **a)** Find the number of subspaces of V of codimension 2.
    - **b)** Let W be a subspace of dimension k. Find the number of hyperplanes of V that contain W.
    - **c)** Let W be a subspace of dimension k. Find the number of subspaces U of V such that $V = W \oplus U$.
8. Let V be a vector space (of any dimension!).
    - **a)** Let X be any subset of V. Show that X contains a maximal linearly independent set.
    - **b)** Let X be any generating subset of V. Show that X contains a basis of V.
9. Let K be a field.
    - **a)** Show that $K[X] \approx K[X^2]$ both as a ring and a vector space, i.e. as a K-algebra.
    - **b)** Find a subspace U of $K[X]$ such that $K[X] = K[X^2] \oplus U$.
10. Let K be a field and $V = K[X]/\langle X^n - 1\rangle$. Let A be the K-subalgebra generated by X^2 (i.e. the subring of V generated by $K \cup \{X^2\}$. When is $V = A$?