Lie Algebras HW1

Nondegenerate Bilinear Symmetric and Skewsymmetric Forms

Ali Nesin
Gümüşlük Akademisi
July 23rd, 2000

Let $V$ be a vector space over a field $F$.

1. Let $A$ and $B$ be subspaces of $V$. Show that $\dim(A + B) + \dim(A \cap B) = \dim(A) + \dim(B)$.

Let $f : V \times V \to F$ be a bilinear map on $V$.

2. Assume $\dim F(V) < \infty$. Let $v_1, \ldots, v_m$ be a basis of $V$. Let $A = (f(v_i, v_j))_{i,j} \in M_{mn}(F)$. Show that $f(v, w) = v^tAw$ for all $v, w \in V$.

We say that $f$ is nondegenerate if $f(v, V) = 0$ implies $v = 0$.

3. Assume $\dim F(V) < \infty$. Show that $f$ is nondegenerate iff the matrix $A$ above is invertible.

Let $v \in V^\prime$. Let $f_v : V \to F$ be given by $f_v(w) = f(v, w)$.

4. Show that $f_v$ is linear.

5. Show that if $f$ is nondegenerate then $f_v$ is onto.

6. What is $\dim(\text{Ker}(f_v))$?

We let $v^\perp = \text{Ker}(f_v) = \{w \in V : f(v, w) = 0\}$ and for $W \leq V$ we let $W^\perp = \{v \in V : f_v(W) = 0\}$.

7. Let $W \leq V$ and assume that $W$ is finite dimensional. Show that $\dim(W) + \dim(W^\perp) \geq \dim(V)$. Hint: Consider the map that sends $v \in V$ to $v^* \in \text{End}(W, F)$ where $v(w) = f(v, w)$ for all $w \in W$.

8. Show that if $f$ is nondegenerate then there is a $w$ such that $f(v, w) = 1$.

9. We assume here that the bilinear nondegenerate form $f$ is skewsymmetric, i.e. that $f(v, v) = 0$ all $v \in V$.

9a. Show that $f(v, w) = -f(w, v)$ for all $v, w \in V$.

If $V = U_1 \oplus U_2$ and $f(U_1, U_2) = 0$, then we write $V = U_1 \perp U_2$. 

9b. Let \( v \) and \( w \) be as in Question #7. Show that \( V = \langle v, w \rangle \perp (v^\perp \cap w^\perp) \)

9c. Assume \( \dim F(V) < \infty \). Let \( A \) be as in Question #2. Show that \( A^t = -A \).

9d. Show that if \( \dim F(V) < \infty \) then \( \dim F(V) \) is even.

9e. Let \( \dim F(V) = 2n \). Find a basis \( v_1, ..., v_n, w_1, ..., w_n \) such that \( f(v_i, v_j) = f(v_i, w_k) = f(w_j, w_k) = 0 \) and \( f(v_i, w_i) = 1 \) for all \( j, k \neq i \).

Let \( A \) be as in question #2 with respect to the above basis. Find \( A \) explicitly.

9f. Let \( sp(V) = \{ \phi \in \text{End}_F(V) = \text{gl}(V) : f(\phi(v), w) = -f(v, \phi(w)) \} \). Show that \( sp(V) \) is a vector space (in fact it is a Lie algebra) over \( F \).

9g. Assuming \( \dim F(V) = 2n \), show that \( sp(V) = \{ X \in M_{2n \times 2n}(F) : X^tA = -AX \} \).

9h. Assuming \( \dim F(V) = 2n \), find \( \dim F(sp(V)) \) and a basis of it.

10. We assume here that the bilinear nondegenerate form \( f \) is symmetric, i.e. that
\[
f(v, w) = f(w, v)
\]
for all \( v, w \in V \).

10a. Show that the matrix \( A \) of question 2 is symmetric, i.e. that \( A^t = A \).

10b. Let \( v \in V \) be such that \( f(v, v) \neq 0 \). Show that \( V = \langle v \rangle \perp v^\perp \).

10c. Assume \( \text{char}(F) \neq 2 \). Show that \( V \) has a basis with respect to which the matrix \( A \) is diagonal. Can we choose a basis so that \( A = \text{Id} \)? What is the number of nonequivalent symmetric nondegenerate bilinear forms on a vector space of dimension \( n \) in terms of \( |F^*/F^*|^2 \) and \( n \)? What about if \( F \) is real-closed, finite, algebraically closed or simply square root closed?

10d. Show that if \( \text{char}(F) = 2 \), then \( V \) has a basis with respect to which the matrix \( A \) is of the form

10e. Let \( f \) be defined by the matrix

\[
A = \text{...}
\]

Show that \( A \) is not equivalent to a diagonal matrix if \( \text{char}(F) = 2 \). Show that if \( \text{char}(F) \neq 2 \) then \( A \) is equivalent to a diagonal matrix. Show that if \( -1 \) is a square in \( F \) then \( A \) is equivalent to \( \text{Id} \).