Lie Algebras HW1

Nondegenerate Bilinear Symmetric and Skewsymmetric Forms

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Let V be a vector space over a field F.

1. Let A and B be subspaces of V. Show that $\dim(A + B) + \dim(A \cap B) = \dim(A) + \dim(B)$.

Let $f : V \times V \to F$ be a bilinear map on V.

2. Assume $\dim_F(V) < \infty$. Let $v_1, ..., v_m$ be a basis of V. Let $A = (f(v_i, v_j))_{i,j} \in M_{n\times n}(F)$. Show that $f(v, w) = v^t A w$ for all $v, w \in V$.

We say that f is **nondegenerate** if $f(v, V) = 0$ implies $v = 0$.

3. Assume $\dim_F(V) < \infty$. Show that f is nondegenerate iff the matrix A above is invertible.

Let $v \in V^\#$. Let $f_v : V \to F$ be given by $f_v(w) = f(v, w)$.

4. Show that f_v is linear.

5. Show that if f is nondegenerate then f_v is onto.

6. What is $\dim(\mathrm{Ker}(f_v))$?

We let $v^\perp = \mathrm{Ker}(f_v) = \{w \in V : f(v, w) = 0\}$ and for $W \le V$ we let $W^\perp = \{v \in V : f_v(W) = 0\}$.

7. Let $W \le V$ and assume that W is finite dimensional. Show that $\dim(W) + \dim(W^\perp) \ge \dim(V)$. **Hint:** Consider the map that sends $v \in V$ to $v^* \in \mathrm{End}(W, F)$ where $v(w) = f(v, w)$ for all $w \in W$.

8. Show that if f is nondegenerate then there is a w such that $f(v, w) = 1$.

9. We assume here that the bilinear nondegenerate form f is **skewsymmetric**, i.e. that

$$f(v, v) = 0 \text{ all } v \in V.$$

9a. Show that $f(v, w) = -f(w, v)$ for all $v, w \in V$.

If $V = U_1 \oplus U_2$ and $f(U_1, U_2) = 0$, then we write $V = U_1 \perp U_2$.

9b. Let v and w be as in Question #7. Show that $V = \langle v, w\rangle \perp (v^\perp \cap w^\perp)$

9c. Assume $\dim_F(V) < \infty$. Let A be as in Question #2. Show that $A^t = -A$.

9d. Show that if $\dim_F(V) < \infty$ then $\dim_F(V)$ is even.

9e. Let $\dim_F(V) = 2n$. Find a basis $v_1,\ldots, v_n, w_1,\ldots, w_n$ such that $f(v_i, v_j) = f(v_i, w_k) = f(w_i, w_j) = 0$ and $f(v_i, w_i) = 1$ for all $j, k \neq i$.

Let A be as in question #2 with respect to the above basis. Find A explicitely.

9f. Let $\mathrm{sp}(V) = \{\varphi \in \mathrm{End}_F(V) = \mathrm{gl}(V) : f(\varphi(v), w) = -f(v, \varphi(w))\}$. Show that $\mathrm{sp}(V)$ is a vector space (in fact it is a Lie algebra) over F.

9g. Assuming $\dim_F(V) = 2n$, show that $\mathrm{sp}(V) \approx \{X \in M_{2n\times 2n}(F) : X^tA = -AX\}$.

9h. Assuming $\dim_F(V) = 2n$, find $\dim_F(\mathrm{sp}(V))$ and a basis of it.

10. We assume here that the bilinear nondegenerate form f is s**ymmetric**, i.e. that

$$f(v, w) = f(w, v) \text{ for all } v, w \in V.$$

10a. Show that the matrix A of question 2 is symmetric, i.e. that $A^t = A$.

10b. Let $v \in V$ be such that $f(v, v) \neq 0$. Show that $V = \langle v\rangle \perp v^\perp$.

10c. Assume $\mathrm{char}(F) \neq 2$. Show that V has a basis with respect to which the matrix A is diagonal. Can we choose a basis so that $A = \mathrm{Id}$? What is the number of nonequivalent symmetric nondegenerate bilinear forms on a vector space of dimension n in terms of $|F^*/F^{*2}|$ and n? What about if F is real-closed, finite, algebraically closed or simply square root closed?

10d. Show that if $\mathrm{char}(F) = 2$, then V has a basis with respect to which the matrix A is of the form

10e. Let f be defined by the matrix

$A =$

Show that A is not equivalent to a diagonal matrix if $\mathrm{char}(F) = 2$. Show that if $\mathrm{char}(F) \neq 2$ then A is equivalent to a diagonal matrix. Show that if -1 is a square in F then A is equivalent to Id.