Linear Algebra Midterm
2009 Spring
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1. *Show that an $n \times n$ matrix is nilpotent if and only if its characteristic polynomial is X^n.*
Proof: Since a matrix is a root of its characteristic polynomial, from right to left is clear. Conversely, suppose that the matrix is nilpotent. We may assume that the base field K is algebraicaly closed. Since the characteristic polynomial is an invariant of the conjugacy class of the matrix, we may also assume that the matrix is lready in the Jordan canonical form. Since the matrix is nilpotent, all the eigenvalues must be 0. (If $M^k = 0$ and $Mv = \lambda v$ for $v \neq 0$ then $0 = M^k v = \lambda^k v$ and so $\lambda = 0$). Let us just consider one Jordan block M of size $k \times k$:

$$M = \begin{pmatrix} 0 & 1 & 0 & \cdots & \vdots \\ 0 & 0 & 1 & \cdots & \vdots \\ 0 & 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

We should compute:

$$\det(M - XId) = \begin{vmatrix} -X & 1 & 0 & \cdots & \vdots \\ 0 & -X & 1 & \cdots & \vdots \\ 0 & 0 & -X & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & 0 & 0 & \cdots & -X \end{vmatrix} = (-1)^k X^k.$$

Since the charactersitic polynomial must be monic, this shows that the characteristic polynomial of each Jordan block of size k is X^k. Now if the Jordan blocks have sizes $k_1, ..., k_r$, since the determinant behaves well with respect to the blocks, the characteristic polynomial of the matrix is $X^{k_1} X^{k_2} \cdots X^{k_r} = X^{k_1 + \cdots k_r} = X^n$.

2. *Show that a square matrix is diagonalizable if and only if its minimal polynomial has no multiple roots in its algebraic closure.*
Proof: ($\Rightarrow$) We may assume again that the matrix is already diagonalizable. If $\lambda_1, ..., \lambda_r$ are the distinct eigenvalues, then the matrix is of the form diag($\lambda_1, ..., \lambda_1, \lambda_2, ..., \lambda_2, ..., \lambda_r, ..., \lambda_r$). It is then clear that the matrix is a root of the polynomial $(X - \lambda_1)(X - \lambda_2) .... (X - \lambda_r)$ and that the matrix is not the root of a smaller factor of that polynomial. Hence the polynomial

$$(X - \lambda_1)(X - \lambda_2) .... (X - \lambda_r)$$

Which has distict roots is the minimal polynomial of the matrix.
($\Leftarrow$) Suppose $\lambda_1, ..., \lambda_r$ are distinct and $(X - \lambda_1)(X - \lambda_2) .... (X - \lambda_r)$ is the minimal polynomial of the matrix. We may assume that the matrix is in Jordan canonical form. We may also assume that the matrix is one Jordan block, say of the form

$$\begin{pmatrix} \lambda & 1 & 0 & \cdots & \vdots \\ 0 & \lambda & 1 & \cdots & \vdots \\ 0 & 0 & \lambda & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & 0 & 0 & \cdots & \lambda \end{pmatrix}.$$

Applying this matrix to the minimal polynomial we get

$$0 = \begin{pmatrix} \lambda-\lambda_1 & 1 & 0 & \cdots & \vdots \\ 0 & \lambda-\lambda_1 & 1 & \cdots & \vdots \\ 0 & 0 & \lambda-\lambda_1 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & 0 & 0 & \cdots & \lambda-\lambda_1 \end{pmatrix} \cdots \begin{pmatrix} \lambda-\lambda_r & 1 & 0 & \cdots & \vdots \\ 0 & \lambda-\lambda_r & 1 & \cdots & \vdots \\ 0 & 0 & \lambda-\lambda_r & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & 0 & 0 & \cdots & \lambda-\lambda_r \end{pmatrix}.$$

Only looking at the diagonal entries of the product we see that $(\lambda - \lambda_1) \dots (\lambda - \lambda_r) = 0$ so that $\lambda = \lambda_i$ for some i. Let v_1 be the first vector of the basis considered for the matrix above. Note that each factor M_i of the product acts on $\langle v_1 \rangle$, so that we can compute modulo $\langle v_1 \rangle$. Note also that modulo $\langle v_1 \rangle$, $M_i(v_2) \equiv (\lambda - \lambda_i)v_2$. Hence applying this product $M_1 \dots M_r$ of matrices to v_2 modulo $\langle v_2 \rangle$ we get:

$$0 = M_1 \dots M_r(v_2) \equiv (\lambda - \lambda_1)(\lambda - \lambda_2)\dots(\lambda - \lambda_r)v_2,$$

A contradiction.

3. *Let K be a field, let A be an $n \times n$ matrix over K. Suppose $A^m = 0$ for some integer m. Show that $A^k = 0$ for some $k \leq n$.*

Proof: By the first part, the characteristic polynomial of A is X^n. Thus the minimal polynomial of A, having to divide the characteristic polynomial, is of the form X^k for some $k \leq n$. Hence $A^k = 0$.

4. *Let φ be a linear transformation whose matrix in some basis is Jordan cell $J_n(\lambda)$. Find all φ-invariant subspaces.*

Answer: Let V be the vector space. Let $v_1, \dots, v_n$ be the Jordan basis. Let $V_i = \langle v_1, \dots, v_i \rangle$. Clearly each V_i is φ-invariant and $V = V_n$. We claim that there are no other φ-invariant subspaces than $0 = V_0, V_1, \dots, V_n$. We will show this by induction on n. Let $0 \neq W \leq V$ be a φ-invariant subspace. Note that W is also $(\varphi - \lambda\mathrm{Id})$-invariant. Since

$$(\varphi - \lambda\mathrm{Id})(\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n) = \alpha_2 v_1 + \dots + \alpha_n v_{n-1},$$

If $0 \neq w = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_i v_i \in W$ with $\alpha_i \neq 0$, then, $(\varphi - \lambda\mathrm{Id})^{i-1} w = \alpha_i v_1 \in W$. Hence $v_1 \in W$ and $V_1 \leq W$.

Now consider V/V_1. φ still acts on V/V_1 and the matrix of this action with respect to the basis $\overline{v_2}, \dots, \overline{v_n}$ is still the Jordan cell $J_{n-1}(\lambda)$. By induction, $W/V_1 = V_i/V_1$ for some i. Since both V_i and W contain V_1, this implies that $W = V_i$.

5. *How to find the Jordan normal forms for the matrices A^2 and A^{-1} if you know the Jordan normal form for an invertible matrix A?*

Answer: Just square/invert the diagonal elements.

6. *Is the linear transformation defined by matrix*

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

diagonalizable? If yes, find a basis in which this transformation has a diagonal matrix, and find this matrix.

Answer: Let $v_1, ..., v_n$ be the basis considered. For $1 \le i \le [n/2]$, since φ interchanges the vectors v_i and v_{n-i+1}, φ acts acts on $V_i = \langle v_i, v_{n-i+1}\rangle$ and its matrix with respect to this basis is

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

If char $K \neq 2$, then $v_i + v_{n-i+1}$ and $v_i - v_{n-i+1}$ are distinct eigen vectors with eigenvalue 1 and -1 respectively. And these give $2[n/2]$ distinct eigenvalues. If $n = 2[n/2]$, i.e. if n is even we see that φ is diagonalizable with entries 1 and -1 evenly distributed. For n odd, $W = \langle v_{[n/2]+1}\rangle$ is already an eigenspace and we get $2[n/2] + 1 = n$ distinct eigenvectors and φ is still diagonalizable with entries 1 and -1, but this time we have one more 1 than -1.
If char $K = 2$ then the characteristic polynomial of the above 2×2 matrix is $X^2 + 1 = (X + 1)^2$, so 1 is the only eigenvalue. Unless $n = 1$, φ is not diagonalizable. But its Jordan Canonical Form is composed of Jordan blocks of the form

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

together with a singleton 1 in case n is odd.

7. *Show that any matrix A such that $A^n = \mathrm{Id}$ for some $n \neq 0$ over an algebraically closed field of characteristic 0 is diagonalizable. Show that this is false if characteristic* > 0. (10 pts.)
Proof: Write A in the Jordan Canonical Form. Let

$$\begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ 0 & 0 & \lambda & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & 0 & 0 & \cdots & \lambda \end{pmatrix}$$

be one of Jordan blocks. Then its n^{th} power is of the form

$$\begin{pmatrix} \lambda^n & n\lambda & * & \cdots & * \\ 0 & \lambda^n & n\lambda & \cdots & 0 \\ 0 & 0 & \lambda^n & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & n\lambda \\ 0 & 0 & 0 & \cdots & \lambda^n \end{pmatrix}$$

which must be Id. So $\lambda^n = 1$ and $n\lambda = 0$. Since char $= 0$, $n \neq 0$, we get a contradiction unless the Jordan block has size 1×1.
Note that this proof wworks as well if characteristic of the field does not divide n.

8. Let φ be the linear transformation defined in some basis by the matrix

$$\begin{pmatrix} -3 & -4 & 0 \\ 2 & 3 & 0 \\ 8 & 8 & 1 \end{pmatrix}$$

8i. The characteristic and minimal polinomial of φ,
8ii. The eigen-values and eigen-vectors of φ,
8iii. The Jordan Canonical form of the matrix.
Proof: Boring.

9. *Find the eigen-values of the operator $f(x) \mapsto f(ax + b)$ in the vector space of real polynomials of degree at most n.*
Proof: Bu soruyu atlamışım. Bunu yapamadım. Çok zor gibi. Çıkarıyorum sınavdan. Belki bunu take home veririz ayrıca. Sizin kolay bir çözümünüz mü var?

10. Let $p(x) = (x - 1)(x - 2)^2(x - 3)^3$.

10i. Find the number of elements in $\text{Mat}_{6\times6}(\mathbb{C})$, up to conjugation, with characteristic polynomial $p(x)$.Give a representative from each conjugacy class.

10ii. Find the number of elements in $\text{Mat}_{6\times6}(\mathbb{C})$, up to conjugation, with minimal polynomial $p(x)$. Give a representative from each conjugacy class.

11. Let $A \leq \text{GL}(V)$ be an abelian group. Suppose that each element of A is diagonalizable. Show that there is a basis of V with respect to which all the elements of A are diagonalizable.