Linear Algebra Final
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1. Let K be a field. We will define a "geometry" $\pi = \pi(K)$ of lines and points as follows: Let ∞ be a new symbol. The "geometry" π will have three sorts of points:

- the elements (x, y) of $K \times K$. These are called "affine points" of π.
- for each $m \in K$, (m) will be a point of K. These points are called "points at infinity".
- (∞) will be a point of π. This is also a "point at infinity".

The lines of π will be

- the non vertical "affine lines" of $K \times K$ of the form
$$[m, b] := \{(x, y) : y = mx + b\} \cup \{(m)\}$$
for all $m, b \in K$.
- the vertical "affine" lines of $K \times K$ of the form
$$[a] := \{(a, y) : y \in K\} \cup \{(\infty)\}$$
for all $m \in K$.
- the line at infinity $[\infty] := \{(m) : m \in K\} \cup \{(\infty)\}$.

a) Show that any two distinct lines intersect at a unique point.
b) Show that through any two distinct points a unique line passes.
c) Show that there are four points such that no three of which are collinear.
d) Assume $K = \mathbf{F}_q$, the field with q elements. Find the number of points on each line, the number of lines through each line and the total number of lines and points. In case $q = 2$, represent the points and the lines of the geometry visually on your exam paper.

2. A collection of subsets (called "lines") of a set P of "points" is called a projective plane if the properties a, b, c above hold.

a) Show that in a projective plane there is a one two one correspondance between the set of points of any two lines.

b) Show that in a projective plane there is a one two one correspondance between the set of lines that pass through any two points.

c) Show that if in a projective plane a line has $n + 1$ points, then through any point $n + 1$ lines pass and the total number of points is $n^2 + n + 1$ and the total number of lines is $n^2 + n + 1$ as well.

d) Show that if in a projective plane we exchange the roles of lines and points, we again obtain a projective plane.

e) Define the concept of isomorphism of projective planes. Define the concept of automorphism of a projective plane.

3. Let K be a field. We will define a "geometry" $\pi' = \pi'(K)$ of lines and points as follows:

The lines of π' will be the subspaces of K^3 of dimension 2 (i.e. planes passing through the origin (0, 0, 0)).

The points of π' will be the subspaces of K^3 of dimension 1 (i.e. lines passing through the origin (0, 0, 0)).

A point of π' will be said to lie on a line of π' if the subspace of dimension 1 that represents the point is a subset of the subspace of dimension 2 that represents the line.

a) Show that π' is a projective plane.

b) Show that any element of $GL_3(K)$ gives rise naturally to an automorphism of π'. Show that the scalars act trivially on π'. Deduce that $PGL_3(K)$ embeds in $Aut(\pi')$.

c) Show that $Aut(K)$ embeds in $Aut(\pi')$.

d) Show that $\pi' \approx \pi(K)$.

e) Find $Aut\ \pi$.