PART I.

1. Let \( f : \mathbb{R}^3 \rightarrow \mathbb{R}^4 \) be defined by \( f(x, y, z) = (x - y, 0, 2x - 2y, x + y - 2z) \).
   
   1.1. Show that \( f \) is a linear map.
   
   1.2. Find a basis of \( \text{Im}(f) \).
   
   1.3. Find a basis of \( \text{Ker}(f) \).

2. Let \( W = \{(x - y, x - y + z, z, 0, 2z) : x, y, z \in \mathbb{R}\} \) be a subspace of \( \mathbb{R}^5 \). Find a basis of the quotient space \( \mathbb{R}^5/W \).

3. Let \( f : V \rightarrow W \) be a linear map between two vector spaces \( V \) and \( W \). Show that if \( v_1, \ldots, v_n \in V \) are such that \( f(v_1), \ldots, f(v_n) \) are linearly independent, then \( v_1, \ldots, v_n \) are also linearly independent.

4. Let \( V \) be a vector space and \( A \) and \( B \) be two subspaces of \( V \). Show that \( A + B = \text{Vect}(A \cup B) \).

5. Let \( V \) and \( W \) be two vector spaces of dimension \( n \) and \( m \) over the same field \( K \). What is the dimension of \( V \times W \)?

6. Let \( V \) be a vector space and \( U \) and \( W \) be two subspaces of \( V \). Show that
   \[ \dim(U + W) + \dim(U \cap V) = \dim U + \dim V. \]
   (Hint: Consider the map \( f : U \times V \rightarrow U + V \) given by \( f(u, v) = u + v \).

PART II

Let \( V \) be a vector space over a field \( K \). Let \( \text{GL}(V) \) denote the group of automorphisms of \( V \). If \( \phi \in \text{GL}(V) \), we say that \( \lambda \in K \) is an eigenvalue of \( \phi \) if \( \phi(v) = \lambda v \) for some \( v \in V^\phi := V \setminus \{0\} \).

1. Let \( \phi \in \text{GL}_K(V) \) have finite order \( n \) and \( \lambda \in K \) be an eigenvalue of \( \phi \). Show that \( \lambda^n = 1 \). Should such a \( \phi \) have to have eigenvalues?

2. Let \( V \) be a vector space over a field \( K \) of characteristic \( p > 0 \). Let \( \phi \in \text{End}_K(V) \).
   
   2a. Show that \( (\phi - 1)^p = \phi^p - 1 \).
   
   2b. Conclude that if \( \phi \) has order \( p^k \) for some \( k > 0 \), then a nonzero vector of \( V \) is fixed by \( \phi \).

A field \( K \) is called algebraically closed if all nonzero polynomials \( f \in K[X] \) have a root in \( K \).
3. Assume \( \dim_K(V) < \infty \) and \( K \) is an algebraically closed field. Let \( A \leq \text{GL}_K(V) \) be an abelian group. Show that the elements of \( A \) have a common nonzero eigenvector. (Hint: By induction on \( \dim V \)).

4. (Schur’s Lemma) Let \( R \) be a ring and \( M \) and \( N \) be two irreducible left \( R \)-modules.

   4a. Show that any homomorphism \( \varphi : M \to N \) is either 0 or an isomorphism.
   4b. Show that \( \text{End}_R(M) \) is a division ring.

5. Assume \( V \) is a vector space of finite dimension over a field \( K \). Let \( A \in \text{End}_K(V) \).

   5a. Show that the subring \( K[A] \) of \( \text{End}_K(V) \) generated by \( A \) and the scalar multiplications \( \lambda \text{Id}_V \) (for \( \lambda \in K \)) is isomorphic to \( K[X]/(f) \) for some polynomial \( f \in K[X] \).
   5b. Can you bound the degree of \( f \) in terms of \( \dim_K(V) \)?
   5c. Find \( f \) when

\[
A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad K = \mathbb{Z}/7\mathbb{Z} \text{ and } A = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}
\]

6. Consider \( \mathbb{Z} \times \mathbb{Z} \) as a group (i.e. as a \( \mathbb{Z} \)-module). For \( A \in \text{End}_\mathbb{Z}(\mathbb{Z} \times \mathbb{Z}) \) consider the subring \( \mathbb{Z}[A] \) of \( \text{End}_\mathbb{Z}(\mathbb{Z} \times \mathbb{Z}) \) generated by \( A \).

   6a. Find the number of minimal generators of \( \mathbb{Z}[A] \) as a \( \mathbb{Z} \)-module when

\[
A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}
\]

   6b. Find the invertible and nilpotent elements of \( \mathbb{Z}[A] \) and its idempotents\(^1\).

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\(^1\) An element \( r \) of a ring is nilpotent if \( r^n = 0 \) for some \( n \) and it is idempotent if \( r^2 = r \).