Math 231 Linear Algebra

Midterm 1
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PART I.

1. Let $f : \mathbb{R}^3 \to \mathbb{R}^4$ be defined by $f(x, y, z) = (x - y, 0, 2x - 2y, x + y - 2z)$.
1.1. Show that f is a linear map.
1.2. Find a basis of Im(f).
1.3. Find a basis of Ker(f).

2. Let $W = \{(x - y, x - y + z, z, 0, 2z) : x, y, z \in \mathbb{R}\}$. W is a subspace of $\mathbf{R}^5$. Find a basis of the quotient space $\mathbb{R}^5/W$.

3. Let $f : V \to W$ be a linear map between two vector spaces V and W. Show that if $v_1, ..., v_n \in V$ are such that $f(v_1), ..., f(v_n)$ are linearly independent, then $v_1, ..., v_n$ are also linearly independent.

4. Let V be a vector space and A and B be two subspaces of V. Show that $A + B =$ Vect$(A \cup B)$.

5. Let V and W be two vector spaces of dimension n and m over the same field K. What is the dimension of $V \times W$.

6. Let V be a vector space and U and W be two subspaces of V. Show that

$$\dim(U + W) + \dim(U \cap V) = \dim U + \dim V.$$

(Hint: Consider the map $f : U \times V \to U + V$ given by $f(u, v) = u + v$.

PART II

Let V be a vector space over a field K. Let GL(V) denote the group of automorphisms of V. If $\varphi \in$ GL(V), we say that $\lambda \in K$ is an eigen value of φ if $\varphi(v) = \lambda v$ for some $v \in V^\# := V \setminus \{0\}$.

1. Let $\varphi \in GL_K(V)$ have finite order n and $\lambda \in K$ be an eigenvalue of φ. Show that $\lambda^n = 1$. Should such a φ have to have eigenvalues?

2. Let V be a vector space over a field K of characteristic $p > 0$. *Let* $\varphi \in \mathrm{End}_K(V)$.

2a. Show that $(\varphi - 1)^{p^k} = \varphi^{p^k} - 1$.

2b. Conclude that if φ has order p^k for some $k > 0$, then a nonzero vector of V is fixed by φ.

A field K is called algebraically closed if all nonzero polynomials $f \in K[X]$ have a root in K.

3. Assume $\dim_K(V) < \infty$ and K is an algebraically closed field. Let $A \leq GL_K(V)$ be an abelian group. Show that the elements of A have a common nonzero eigenvector. (Hint: By induction on dim V).

4. **(Schur's Lemma)** Let R be a ring and M and N be two irreducible left R-modules.

4a. Show that any homomorphism $\varphi : M \to N$ is either 0 or an isomorphism.

4b. Show that $\text{End}_R(M)$ is a division ring.

5. Assume V is a vector space of finite dimension over a field K. Let $A \in \text{End}_K(V)$.

5a. Show that the subring $K[A]$ of $\text{End}_K(V)$ generated by A and the scalar multiplications λId_V (*for* $\lambda \in K$) is isomorphic to $K[X]/\langle f \rangle$ for some polynomial $f \in K[X]$.

5b. Can you bound the degree of f in terms of $\dim_K(V)$?

5c. Find f when

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$K = \mathbb{Z}/7\mathbb{Z} \text{ and } A = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$$

6. Consider $\mathbb{Z} \times \mathbb{Z}$ as a group (i.e. as a $\mathbb{Z}$-module). For $A \in \text{End}_{\mathbb{Z}}(\mathbb{Z} \times \mathbb{Z})$ consider the subring $\mathbb{Z}[A]$ of $\text{End}_{\mathbb{Z}}(\mathbb{Z} \times \mathbb{Z})$ generated by A.

6a. Find the number of minimal generators of $\mathbb{Z}[A]$ as a $\mathbb{Z}$-*module when*

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

6b. Find the invertible and nilpotent elements of $\mathbb{Z}[A]$ and its idempotents[1].

[1] An element r of a ring is nilpotent if $r^n = 0$ for some n and it is idempotent if $r^2 = r$.