Algebra Quiz

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1. Let G be a group and K a field. In this and the next exercise, it is advised to write G multiplicatively. Consider the formal elements of the form

$$\sum_{g\in G}\lambda_g g$$

where $\lambda_g \in K$ and only finitely many of them are nonzero. Let $K[G]$ be the set of such elements.

1a. Find the elements of $\mathbf{F}_2[\mathbb{Z}/3\mathbb{Z}]$. (2 pts.)

Define +, × and scalar multiplication formally on $K[G]$ as follows:

$$\left(\sum_{g\in G}\lambda_g g\right)+\left(\sum_{g\in G}\mu_g g\right)=\sum_{g\in G}(\mu_g+\lambda_g)g$$

$$\left(\sum_{g\in G}\lambda_g g\right)\times\left(\sum_{g\in G}\mu_g g\right)=\sum_{g\in G}\left(\sum_{hk=g}\mu_h\lambda_k\right)g$$

$$\lambda\left(\sum_{g\in G}\lambda_g g\right)=\sum_{g\in G}\lambda\lambda_g g$$

Then $K[G]$ becomes a (not necessarily commutative) ring with 1 and also a K-vector space satisfying $\lambda(ab) = (\lambda a)b = a(\lambda b)$ (such a structure is called an **algebra** or a ***K*-algebra**, e.g. $\mathrm{End}_K(V)$ is a K-algebra).

1b. Show that $G \le K[G]^*$. (2 pts.)

1c. Find the invertible and the nilpotent elements and the idempotents of $\mathbf{F}_2[\mathbf{Z}/3\mathbf{Z}]$. (4 pts.)

1d. If G is finite what is $\left(\sum_{g\in G} g\right)^2$? (4 pts.)

1e. Show that if G has torsion elements, then $K[G]$ has zero-divisors. (2 pts.)

1f. Show that $K[\mathbb{Z}]$ has no zero-divisors. (4 pts.)

1g. Show that the set of elements of the form $\sum_{g\in G}\lambda_g g$ where $\sum_{g\in G}\lambda_g = 0$ forms an ideal of $K[G]$. (3 pts.)

1h. Let G be a group, K a field and $\varphi : G \to \mathrm{GL}(V) \subseteq \mathrm{End}_K(V)$ a group homomorphism. Show that φ extends uniquely to a K-algebra homomorphism $\underline{\varphi} : K[G] \to \mathrm{End}_K(V)$. (3 pts.)

1i. Notation as above. Show that defining av as $\varphi(a)(v)$ for $a \in K[G]$ and $v \in G$, V becomes a $K[G]$-module via $\underline{\varphi}$. (1 pt.)

2. The purpose of this exercise is to prove **Maschke's Theorem** that states the following: Let G be a finite group, K a field whose characteristic does not divide $|G|$ and V a $K[G]$-module. Then V is completely reducible, i.e. any submodule of V has a complement in V.

2a. Show that a vector space endomorphism u of V is a $K[G]$-module endomorphism iff $u(gv) = gu(v)$ for all $g \in G$ and $v \in V$. (3 pts.)

2b. Let W be a $K[G]$-submodule of V. Let U be a complement of W in V (as a vector space over K). Thus $V = W \oplus U$. Let π be the projection of V onto W according to this decomposition. Let $u : V \to V$ be defined by $u(v) = \sum_{g\in G} g\,\pi(g^{-1}v)$. Show that $u(V) \le W$, that u is a $K[G]$-module homomorphism, that in case G is finite $u_{|W} = |G|\,\mathrm{Id}_W$ and that $u \circ u = |G|\,u$. (10 pts.)

2c. Assume now that G is finite and that char(K) does not divide $|G|$. Let $v = \frac{1}{|G|}u$. Show that $V = W \oplus \text{Ker}(v)$. (Now Ker($v$) is a $K[G]$-module.) (8 pts.)

2d. Show that if further $\dim_K(V) < \infty$ then V is a direct sum of irreducible modules. (5 pts.)