1. Let G be a group. The equation $(xy)^n = x^n y^n$ holds for all $x, y \in G$ if and only if G is abelian.

2. Describe the symmetry group of an isosceles triangle that is not equilateral.

3. Find a group G and subgroups H and K of G such that $H \trianglelefteq K$ and $K \trianglelefteq G$ and $H \ntrianglelefteq G$.

4. Find all groups which have exactly three subgroups.

5. $\mathbb{Q}$ is not finitely generated.

6. Let G be a group, and let G^G be the set of all functions from G into G. For α, $\beta \in G^G$, define $\alpha + \beta \in G^G$ by declaring $x^{\alpha+\beta} := x^\alpha x^\beta$ for all $x \in G$. Is G^G a ring under this addition and the composition of maps ?

7. Let G, G^G and $+$ be as in the previous question. If α and $\beta \in G^G$ are homomorphisms, what is the necessary and sufficient condition for $\alpha + \beta$ to be a homomorphism ?

8. Let $C_n = \langle g \rangle$ be a cyclic group of order $n \in N$. A homomorphism $\varphi : C_n \longrightarrow H$, where H is an arbitrary group, is completely determined by $g\varphi$. A homomorphism $\varphi : C_n \longrightarrow C_n$ is an automorphism if and only if $o(g\varphi) = n$. Prove that $\mathrm{Aut}(C_n) \cong \mathbb{Z}_n^\times$.

9. Let $S_3 = \langle a, b, c \rangle$ be the symmetric group on 3 letters, $a := (12)$, $b := (13)$, $c := (23)$. A homomorphism $\varphi : S_3 \longrightarrow H$, where H is an arbitrary group, is completely determined by $a\varphi$, $b\varphi$ and $c\varphi$. What are the necessary and sufficient conditions for a homomorphism $\varphi : S_3 \longrightarrow S_3$ to be an automorphism ? Prove that $\mathrm{Aut}(S_3) \cong S_3$.

10. Prove that $\mathrm{Aut}(V_4) \cong S_3$ and $\mathrm{Aut}(D_8) \cong D_8$.

11. Let $\langle a \rangle$ and $\langle b \rangle$ be cyclic groups of order 2. The free product $F = \langle a \rangle * \langle b \rangle$ is an infinite dihedral group.

12. Suppose that a finite group G acts on a set X, and put $\mathrm{Fix}_X(g) := \{x \in X : xg = x\}$. Then the number of orbits is given by the formula $\frac{1}{|G|} \sum_{g \in G} \mathrm{Fix}_X(g)$.

13. If $\sigma = (12)(34) \in S_4$ and $\tau = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{smallmatrix}\right) \in S_4$, what is σ^τ ? Generalize. Describe the conjugacy classes of S_4.

14. Describe the conjugacy classes of S_5, of S_n. How many elements are there in each class ?

15. Find the conjugacy classes of A_5 and the number of elements in each class. Conclude that A_5 is a simple group.

16. Let p, q, r be distinct prime numbers. If G is a group of order pqr, then G is not simple.

17. Let G be a finite group, H a subgroup of G. Let P be a Sylow p-subgroup of G.

If $H \trianglelefteq G$, then $H \cap P$ is a Sylow p-subgroup of H. If $H \ntrianglelefteq G$, then $H \cap P$ is not necessarily a Sylow p-subgroup of H. Any Sylow p-subgroup of H has the form $H \cap P'$, where P' is a Sylow p-subgroup of G.