1. Let $G$ be a group. The equation $(xy)^n = x^ny^n$ holds for all $x, y \in G$ if and only if $G$ is abelian.

2. Describe the symmetry group of an isosceles triangle that is not equilateral.

3. Find a group $G$ and subgroups $H$ and $K$ of $G$ such that $H \subseteq K$ and $K \subseteq G$ and $H \not\supseteq G$.

4. Find all groups which have exactly three subgroups.

5. $\mathbb{Q}$ is not finitely generated.

6. Let $G$ be a group, and let $G^G$ be the set of all functions from $G$ into $G$. For $\alpha, \beta \in G^G$, define $\alpha + \beta \in G^G$ by declaring $x^{\alpha + \beta} := x^\alpha x^\beta$ for all $x \in G$. Is $G^G$ a ring under this addition and the composition of maps?

7. Let $G$, $G^G$ and $+$ be as in the previous question. If $\alpha$ and $\beta \in G^G$ are homomorphisms, what is the necessary and sufficient condition for $\alpha + \beta$ to be a homomorphism?

8. Let $C_n = \langle g \rangle$ be a cyclic group of order $n \in \mathbb{N}$. A homomorphism $\varphi : C_n \rightarrow H$, where $H$ is an arbitrary group, is completely determined by $g\varphi$. A homomorphism $\varphi : C_n \rightarrow C_n$ is an automorphism if and only if $o(g\varphi) = n$. Prove that $\text{Aut}(C_n) \cong \mathbb{Z}_n^\times$.

9. Let $S_3 = \langle a, b, c \rangle$ be the symmetric group on 3 letters, $a := (12)$, $b := (13)$, $c := (23)$. A homomorphism $\varphi : S_3 \rightarrow H$, where $H$ is an arbitrary group, is completely determined by $a\varphi$, $b\varphi$ and $c\varphi$. What are the necessary and sufficient conditions for a homomorphism $\varphi : S_3 \rightarrow S_3$ to be an automorphism? Prove that $\text{Aut}(S_3) \cong S_3$.

10. Prove that $\text{Aut}(V_4) \cong S_3$ and $\text{Aut}(D_8) \cong D_8$.

11. Let $\langle a \rangle$ and $\langle b \rangle$ be cyclic groups of order 2. The free product $F = \langle a \rangle * \langle b \rangle$ is an infinite dihedral group.

12. Suppose that a finite group $G$ acts on a set $X$, and put $\text{Fix}_X(g) := \{x \in X : xg = x\}$. Then the number of orbits is given by the formula $\frac{1}{|G|} \sum_{g \in G} \text{Fix}_X(g)$.

13. If $\sigma = (12)(34) \in S_4$ and $\tau = (1 2 3 4) \in S_4$, what is $\sigma^\tau$? Generalize. Describe the conjugacy classes of $S_4$.

14. Describe the conjugacy classes of $S_5$, of $S_n$. How many elements are there in each class?

15. Find the conjugacy classes of $A_5$ and the number of elements in each class. Conclude that $A_5$ is a simple group.

16. Let $p$, $q$, $r$ be distinct prime numbers. If $G$ is a group of order $pqr$, then $G$ is
not simple.

17. Let $G$ be a finite group, $H$ a subgroup of $G$. Let $P$ be a Sylow $p$-subgroup of $G$.

If $H \trianglelefteq G$, then $H \cap P$ is a Sylow $p$-subgroup of $H$. If $H \ntrianglelefteq G$, then $H \cap P$ is not necessarily a Sylow $p$-subgroup of $H$. Any Sylow $p$-subgroup of $H$ has the form $H \cap P'$, where $P'$ is a Sylow $p$-subgroup of $G$. 