Linear Algebra

Midterm
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1. True or false? (You have to explain the reason of your answer) (8 pts.)

i. $\{(x, y, z) \in \mathbb{R}^3 : x + y - z = xyz\}$ is a subspace of $\mathbb{R}^3$.

ii. $\{(x, y, z) \in \mathbb{R}^3 : xyz \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^3$.

iii. $\{(x, y, z) \in \mathbb{R}^3 : xyz \in \mathbb{Q}\}$ is a vector space over $\mathbb{R}$.

iv. $\{(x, y, z) : x + y - z \in \mathbb{Q}\}$ is a vector space over $\mathbb{Q}$.

v. $\{(x, 2y, 3z) : x, y, z \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^3$.

vi. $\{(x, y, z) \in \mathbb{R}^3 : xy = z\}$ is a subspace of $\mathbb{R}^3$.

vii. $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 - z^2 \geq 0\}$ is a subspace of $\mathbb{R}^3$.

viii. $\{(x + y, y^2, y) : x, y, z \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^3$.

2. True or false? (Explain) (8 pts.)

i. If every proper subspace of a vector space has finite dimension then the vector space has finite dimension.

ii. A subspace of a subspace of a vector space V is a subspace of V.

iii. $\mathbb{R}^2$ has infinitely many subspaces of dimension 1.

iv. $\mathbb{R}^2$ has finitely many subspaces of dimension 2.

v. The only vector space with finitely many subspaces is the trivial vector space $\{0\}$.

vi. If A and B are subspaces of the vector space V, then $A \cap B$ is also a subspace of V.

vii. The vectors v, w and $v + w$ of a vector space can never be linearly independent.

viii. The vectors $v - w$ and $v + w$ of a vector space over $\mathbb{R}$ are linearly independent.

3. Let $V \neq \{0\}$ be a vector space over $\mathbb{R}$. Show that there is a proper subspace $W < V$ and a vector $v \in V$ such that

$$V = \{w + \lambda v : w \in W, \lambda \in \mathbb{R}\}.$$

(10 pts.)

4. Find a basis of the subspace of $\mathbb{R}^5$ generated by the vectors
(1, 2, 3, -1, 0), (2, -1, 0, 1, 1), (8,1, 6, 1, 3), (3, 1, 3, 0, 1).
(8 pts.)

5. Let $V = \{(x, y, z, t, u) \in \mathbb{R}^5 : x + y + z - t + u = 0\}$ and

$W = \{(x, y, z, t, u) \in V : 2x - 2y + z - t + u = 0 \text{ and } 2x - 3t = 0\}$.

i. Find bases of V and W. (4 + 6 pts.)

ii. Find a subspace S such that $V + S = \mathbb{R}^5$. (8 pts.)

iii. Find a subspace U such that $W + U = V$ and $W \cap U = \{0\}$. (8 pts.)

6. Let U and V be subspace of a vector space

i. Show that the set $U + V$ defined by

$$\{u + v : u \in U, v \in V\}$$

is also a subspace. (8 pts.)

ii. Suppose that dim U and dim V are finite. Show that dim $(U + V) \leq$ dim U + dim V. (12 pts.)

iii. Give an example where dim $(U + V) <$ dim U + dim V. (8 pts.)

7. Show that a vector space of dim n over $\mathbb{C}$ can be regarded as a vector space of dimension $2n$ over $\mathbb{R}$. (12 pts.)