Math 231 (Linear Algebra)

Final
January 17, 2004
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I. Let K be a field, A a set and Π a set of subsets of A. Let V be the set of all functions from A into K. Clearly V is a vector space over K with the usual operations (addition and scalar multiplication). Let $V(\Pi)$ be the set of elements of V that vanish on some subset that belongs to Π. Thus,

$$V(\Pi) = \{ f : A \to K : \text{there is } X \in \Pi \text{ such that } f = 0 \text{ on } X\}.$$

Find the necessary and sufficent condition on Π for $V(\Pi)$ to be a subspace of V. (10 pts.)

II. Let $\varphi : \mathbb{R}^2 \to \mathbb{R}^3$ be given by $\varphi(x, y) = (x - y, 2x, y)$. Let

$e_1 = (1, 2)$, $e_2 = (3, 1)$.

$f_1 = (1, 1, 1)$, $f_2 = (1, 0, -1)$, $f_3 = (0, 1, 1)$.

Find the matrix of φ with respect to these bases. (10 pts.)

III. Let V be a vector space over a field F. Let $\varphi : V \to V$ be a linear map. A nonzero vector $v \in V$ is called an **eigenvector** of φ if $\varphi(v) = \alpha v$ for some $\alpha \in F$. Such a scalar α is called an **eigenvalue** of φ. For $\alpha \in F$ we let $V_\alpha = \{v \in V : \varphi(v) = \alpha v\}$.

III.1. Show that V_α is a subspace of V. (2 pts.)

III.2. Find all the eigenvalues and the corresponding eigenvectors of the linear map $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ given by the matrix $\begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}$. (10 pts.)

III.3. Let V be the vector space of real sequences and let φ be the linear map from V into V defined by $\varphi(x_0, x_1, x_2, \ldots) = (x_1, x_2, x_3, \ldots)$. Find the eigenvalues and eigenvectors of φ. (4 pts.)

III.4. Let $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ be a rotation (around (0,0) of course, otherwise φ is not linear). Can φ have an eigenvalue? (10 pts.)

III.5. Assume V is finite dimensional. Show that α is an eigenvalue for φ if and only if α is a root of the polynomial $\det(\varphi - X\mathrm{Id}_V) = 0$. (Here I assume that you know that a linear map from a finite dimensional vector space into itself is invertible iff its determinant is nonzero). Check this result on Question 2. Conclude that a linear map $\mathbb{R}^3$ into itself has always an eigenvector. (10 pts.)

III.6. Show that if $(\alpha_i)_i$ are all distinct scalars and $0 \neq v_i \in V_{\alpha_i}$, then the set $(v_i)_i$ is a linearly independent set. In other words, show that the subspace spanned by the subspaces V_α is a direct sum of them (10 pts.)

III.7. Assume that V is finite dimensional. Show that $V = \oplus_i V_{\alpha_i}$ if and only if there is a basis of V in which the matrix of φ is diagonal. (10 pts.)

III.8. Find a basis of $\mathbb{R}^2$ in which the matrix of the linear map φ in Question III.2 is diagonal. (10 pts.).