Math 217 Linear Algebra
Final Exam
Fabruary 4, 2006

1. Prove or disprove: For the nonzero matrices A, B and C, if $AB = AC$ then $B = C$. (3 pts.)
2. Prove or disprove: For matrices A and B, if B is the inverse of A^2 then AB is the inverse of A. (3 pts.)
3. Find the inverse of the matrix $\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$. (6 pts.)
4. Show that the inverse of the $n \times n$ matrix $\begin{pmatrix} n & -1 & \cdots & -1 \\ -1 & n & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & n \end{pmatrix}$ is of the form

   $\frac{1}{n+1}\begin{pmatrix} c & 1 & \cdots & 1 \\ 1 & c & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & c \end{pmatrix}$ for some c. (6 pts.)
5. Find A^{100} and A^{285} if $A = \begin{pmatrix} 7 & 12 \\ -4 & -7 \end{pmatrix}$. (20 pts.)
6. For what values of $a \in \mathbb{R}$, does the system,

   $ax - 3y + (a-2)z = 3$
   $2ax - 5y + z = 7$
   $au - y - 3z = 5$

has

a) a unique solution in $\mathbb{R}^3$,
b) a solution in $\mathbb{Z}^3$.
c) no solution at all,
d) infinitely many solutions?

For cases a, b and c find all possible solutions. (20 pts.)

7. a) Find all 2×2 matrices A such that $A^2 = A$. (10 pts.)

   b) Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map such that $f^2 = f$. Show that $\text{Im} f \cap \text{Ker} f = 0$. Conclude that if $n = 2$ and $f \neq \text{Id}$ (the identity map) and $f \neq 0$ (the zero map), then dim Ker $f = 1$ and dim Im $f = 1$. (10 pts.)

   c) Show that if A is a 2×2 matrix such that $A^2 = A$, then there is an invertible P such that $PAP^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. (10 pts.)

   d) Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map such that $f^2 = f$. Show that $f(x) - x \in$ Ker f for all $f \in \mathbb{R}^n$. Conclude that $\mathbb{R}^n$ = Ker f + Im f. Deduce that if A is an $n \times n$ matrix such that $A^2 =$

A, then there is an invertible P such that $PAP^{-1} = \begin{pmatrix} \mathrm{Id}_m & 0_{m,n-m} \\ 0_{n-m,m} & 0_{m,m} \end{pmatrix}$ for some $m = 0, 1, \ldots, n$. (Here 0 means the zero matrix of the appropriate size). (22 pts.)