Math 231 (Linear Algebra)
Final
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I. Let $K$ be a field, $A$ a set and $\Pi$ a set of subsets of $A$. Let $V$ be the set of all functions from $A$ into $K$. Clearly $V$ is a vector space over $K$ with the usual operations (addition and scalar multiplication). Let $V(\Pi)$ be the set of elements of $V$ that vanish on some subset that belongs to $\Pi$. Thus,

$$V(\Pi) = \{ f : A \to K : \text{there is } X \in \Pi \text{ such that } f = 0 \text{ on } X \}.$$ 

Find the necessary and sufficient condition on $\Pi$ for $V(\Pi)$ to be a subspace of $V$. (10 pts.)

Solution. $V(\Pi)$ is always closed under scalar multiplication. All we need is to find necessary and sufficient condition(s) on $\Pi$ for $V(\Pi)$ to be closed under addition.

Assume $V(\Pi)$ is closed under addition. Let $X$ and $Y$ be two subsets of $A$. Let $f$ be the characteristic function of $X$. Thus $f(x) = 0$ if $x \in X$ and $f(x) = 1$ if $x \notin X$. Then $f \in V(\Pi)$. Let $g$ be such that $g(x) = 0$ if $x \in X \cup (X^c \cap Y)$ and $g(x) = 1$ otherwise. Then $f + g$ is the characteristic function of $X \cap Y$. On the other hand, since $f + g \in V(\Pi)$, there is a $Z \in \Pi$ such that $f + g = 0$ on $Z$. We must have $Z \subseteq X \cap Y$. Thus for all $X, Y \in \Pi$ there is a $Z \in \Pi$ such that $Z \subseteq X \cap Y$.

It is clear that this condition is also sufficient for $V(\Pi)$ to be closed under addition.

II. Let $\varphi : \mathbb{R}^2 \to \mathbb{R}^3$ be given by $\varphi(x, y) = (x - y, 2x, y)$. Let $e_1 = (1, 2)$, $e_2 = (3, 1)$, $f_1 = (1, 1, 1)$, $f_2 = (1, 0, -1)$, $f_3 = (0, 1, 1)$.

Find the matrix of $\varphi$ with respect to these bases. (10 pts.)

Solution. Note that

$$(1, 0, 0) = f_1 - f_3$$
$$(0, 1, 0) = -f_1 + f_2 + 2f_3$$
$$(0, 0, 1) = f_1 - f_2 - f_3$$

Using these or computing directly, we get

$$\varphi(e_1) = (1 - 2, 2, 2) = (-1, 2, 2) = 3f_3 - f_1$$
$$\varphi(e_2) = (3 - 1, 6, 1) = (2, 6, 1) = -3f_1 + 5f_2 + 9f_3$$

So that the matrix is:

$$
\begin{pmatrix}
-1 & -3 \\
0 & 5 \\
3 & 9
\end{pmatrix}
$$

III. Let $V$ be a vector space over a field $F$. Let $\varphi : V \to V$ be a linear map. A nonzero vector $v \in V$ is called an eigenvector of $\varphi$ if $\varphi(v) = \alpha v$ for some $\alpha \in F$. Such a scalar $\alpha$ is called an eigenvalue of $\varphi$. For $\alpha \in F$ we let $V_\alpha = \{ v \in V : \varphi(v) = \alpha v \}$.

III.1. Show that $V_\alpha$ is a subspace of $V$. (2 pts.)

Proof. This is easy.
III.2. Find all the eigenvalues and the corresponding eigenvectors of the linear map $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ given by the matrix $\begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}$. (10 pts.)

Solution. We have to find solutions of $\begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \alpha \begin{pmatrix} x \\ y \end{pmatrix}$, i.e. of the system

$$2x + 2y = \alpha x$$
$$x + 3y = \alpha y$$
equivalently of the system

$$(2-\alpha)x + 2y = 0$$
$$x + (3-\alpha)y = 0$$

For this system to have a nonzero solution $(x, y)$, we need

$$\det \begin{pmatrix} 2-\alpha & 2 \\ 1 & 3-\alpha \end{pmatrix} = 0,$$

(because otherwise the linear map defined by the matrix is invertible and has trivial kernel) implying $\alpha^2 - 5\alpha + 4 = 0$, i.e. $\alpha = 1$ or 4.

If $\alpha = 1$, the system is equivalent to $x + 2y = 0$.
If $\alpha = 4$, the system is equivalent to $x - y = 0$.
Thus

$$V_1 = \{(x, y) : x + 2y = 0\} = \mathbb{R} (-2, 1)$$
$$V_4 = \{(x, y) : x - y = 0\} = \mathbb{R} (1, 1)$$
$$V_\alpha = 0 \text{ if } \alpha \neq 1, 4.$$ In short 1 and 4 are the two eigenvalues and $(-2, 1)$ and $(1, 1)$, or their nonzero multiples, are the corresponding eigenvectors.

III.3. Let $V$ be the vector space of real sequences and let $\varphi$ be the linear map from $V$ into $V$ defined by $\varphi(x_0, x_1, x_2, \ldots) = (x_1, x_2, x_3, \ldots)$. Find the eigenvalues and eigenvectors of $\varphi$. (4 pts.)

Solution. We need to solve $\alpha(x_0, x_1, x_2, \ldots) = \varphi(x_0, x_1, x_2, \ldots) = (x_1, x_2, x_3, \ldots)$, i.e.

$$x_1 = \alpha x_0$$
$$x_2 = \alpha x_1$$
$$x_3 = \alpha x_2$$
$$x_4 = \alpha x_3$$
$$\ldots$$

Any $\alpha$ is an eigenvalue. The vector $(1, \alpha, \alpha^2, \alpha^3, \ldots)$ is an eigenvector for $\alpha$.

III.4. Let $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ be a rotation (around $(0,0)$ of course, otherwise $\varphi$ is not linear). Can $\varphi$ have an eigenvalue? (10 pts.)

Answer. No, unless it is a rotation of $\pi$ or $2\pi$ radians, in which cases the eigenvalues are $-1$ and 1 respectively. Why? Well, how can you rotate a vector and still get a scalar multiple of that vector?
III.5. Assume $V$ is finite dimensional. Show that $\alpha$ is an eigenvalue for $\varphi$ if and only if $\alpha$ is a root of the polynomial $\det(\varphi - X\text{Id}) = 0$. (Here I assume that you know that a linear map from a finite dimensional vector space into itself is invertible iff its determinant is nonzero). Check this result on Question 2. Conclude that a linear map $\mathbb{R}^3$ into itself has always an eigenvector. (10 pts.)

**Proof:** Let $\alpha$ be an eigenvalue for $\varphi$. Then there is a nonzero vector such that $\varphi(v) = \alpha v$. Hence $\varphi(v) = \alpha v$. It follows that $v \in \text{Ker}(\varphi - \alpha \text{Id})$ and $\varphi - \alpha \text{Id}$ is noninvertible. Let $v$ be a nonzero vector in $\text{Ker}(\varphi - \alpha \text{Id})$. Hence $(\varphi - \alpha \text{Id})(v) = 0$ and $\varphi(v) = \alpha v$. Since $v \neq 0$, this shows that $\alpha$ is an eigenvalue of $\varphi$.

This is exactly how Question 2 was solved.

If $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$, then the polynomial $\det(\varphi - X \text{Id})$ is of degree 3, and must have a solution.

III.6. Show that if $(\alpha_i)_i$ are all distinct scalars and $0 \neq v_i \in V_{\alpha_i}$, then the set $(v_i)_i$ is a linearly independent set. In other words, show that the subspace spanned by the subspaces $V_{\alpha}$ is a direct sum of them (10 pts.)

**Proof:** Assume $\beta_{i_1}, \beta_{i_2}, ..., \beta_{i_n}$ are nonzero scalars and that

$$\beta_{i_1} v_{i_1} + \beta_{i_2} v_{i_2} + ... + \beta_{i_n} v_{i_n} = 0.$$

Applying $\varphi$ to this equality we get

$$\beta_{i_1} \alpha_{i_1} v_{i_1} + \beta_{i_2} \alpha_{i_2} v_{i_2} + ... + \beta_{i_n} \alpha_{i_n} v_{i_n} = 0.$$

Subtract the second equation from the first equation multiplied by $\alpha_{i_1}$ to get

$$\beta_{i_2} (\alpha_{i_1} - \alpha_{i_2}) v_{i_2} + ... + \beta_{i_n} (\alpha_{i_1} - \alpha_{i_n}) v_{i_n} = 0.$$

Now we have $\leq n - 1$ terms. By induction we can conclude that

$$\beta_{i_2} (\alpha_{i_1} - \alpha_{i_2}) = ... = \beta_{i_n} (\alpha_{i_1} - \alpha_{i_n}) = 0.$$

Since $\beta_{i_1}, \beta_{i_2}, ..., \beta_{i_n}$ are nonzero scalars, we get,

$$\alpha_{i_1} - \alpha_{i_2} = ... = \alpha_{i_1} - \alpha_{i_n} = 0,$$

i.e.

$$\alpha_{i_1} = \alpha_{i_2} = ... = \alpha_{i_n}.$$

Hence $n = 1$ and the result is trivial in this case.

III.7. Assume that $V$ is finite dimensional. Show that $V = \oplus_i V_{\alpha_i}$ if and only if there is a basis of $V$ in which the matrix of $\varphi$ is diagonal. (10 pts.)

**Proof:** Trivial.

Suppose there is a basis of $V$ in which the matrix of $\varphi$ is diagonal. Let $(v_i)_i$ be this basis. Then $\varphi(v_i) = \alpha_i v_i$ for some scalar $\alpha_i$. (The $\alpha_i$ is the scalar that appears on the $i$-th column of the diagonal matrix). It is now easy to check that $V = \oplus_i V_{\alpha_i}$. 

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Conversely, suppose $V = \bigoplus_i V_{\alpha_i}$. Choose a basis of each $V_{\alpha_i}$ and collect them together to get a basis of $V$. The matrix of $\varphi$ with respect to this basis must be diagonal.

**III.8.** *Find a basis of $\mathbb{R}^2$ in which the matrix of the linear map $\varphi$ in Question III.2 is diagonal.* (10 pts.).

**Solution.** Take $v_1 = (-2, 1)$ and $v_4 = (1, 1)$. Then the matrix of $\varphi$ with respect to this basis is

$$
\begin{pmatrix}
4 & 0 \\
0 & 1
\end{pmatrix}.
$$