Math 231 (Linear Algebra)

Final
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I. *Let K be a field, A a set and Π a set of subsets of A. Let V be the set of all functions from A into K. Clearly V is a vector space over K with the usual operations (addition and scalar multiplication). Let $V(\Pi)$ be the set of elements of V that vanish on some subset that belongs to Π. Thus,*

$$V(\Pi) = \{ f : A \to K : \text{there is } X \in \Pi \text{ such that } f = 0 \text{ on } X\}.$$

Find the necessary and sufficent condition on Π for $V(\Pi)$ to be a subspace of V. (10 pts.)

Solution. $V(\prod)$ is always closed under scalar multiplication. All we need is to find necessary and sufficient condition(s) on $\prod$ for $V(\prod)$ to be closed under addition.

Assume $V(\prod)$ is closed under addition. Let X and Y be two subsets of A. Let f be the characteristic function of X^c. Thus $f(x) = 0$ if $x \in X$ and $f(x) = 1$ if $x \notin X$. Then $f \in V(\prod)$. Let g be such that $g(x) = 0$ if $x \in X \cup (X^c \cap Y^c)$ and $g(x) = 1$ otherwise. Then $f + g$ is the characteristic function of $X \cap Y$. On the other hand, since $f + g \in V(\prod)$, there is a $Z \in \prod$ such that $f + g = 0$ on Z. We must have $Z \subseteq X \cap Y$. Thus for all $X, Y \in \prod$ there is a $Z \in \prod$ such that $Z \subseteq X \cap Y$.

It is clear that this condition is also sufficient for $V(\prod)$ to be closed under addition.

II. *Let $\varphi : \mathbb{R}^2 \to \mathbb{R}^3$ be given by $\varphi(x, y) = (x - y, 2x, y)$. Let*

$e_1 = (1, 2)$, $e_2 = (3, 1)$.

$f_1 = (1, 1, 1)$, $f_2 = (1, 0, -1)$, $f_3 = (0, 1, 1)$.

Find the matrix of φ with respect to these bases. (10 pts.)

Solution. Note that

$(1, 0, 0) = f_1 - f_3$

$(0, 1, 0) = -f_1 + f_2 + 2f_3$

$(0, 0, 1) = f_1 - f_2 - f_3$

Using these or computing directly, we get

$\varphi(e_1) = (1 - 2, 2, 2) = (- 1, 2, 2) = 3f_3 - f_1$

$\varphi(e_2) = (3 - 1, 6, 1) = (2, 6, 1) = -3f_1 + 5f_2 + 9f_3$

So that the matrix is:

$$\begin{pmatrix} -1 & -3 \\ 0 & 5 \\ 3 & 9 \end{pmatrix}$$

III. *Let V be a vector space over a field F. Let $\varphi : V \to V$ be a linear map. A nonzero vector $v \in V$ is called an* ***eigenvector*** *of φ if $\varphi(v) = \alpha v$ for some $\alpha \in F$. Such a scalar α is called an* ***eigenvalue*** *of φ. For $\alpha \in F$ we let $V_\alpha = \{v \in V : \varphi(v) = \alpha v\}$.*

III.1. *Show that V_α is a subspace of V.* (2 pts.)

Proof. This is easy.

III.2. *Find all the eigenvalues and the corresponding eigenvectors of the linear map* $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ *given by the matrix* $\begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}$. (10 pts.)

Solution. We have to find solutions of $\begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \alpha\begin{pmatrix} x \\ y \end{pmatrix}$, i.e. of the system

$$2x + 2y = \alpha x$$
$$x + 3y = \alpha y$$

equivalently of the system

$$(2-\alpha)x + 2y = 0$$
$$x + (3-\alpha)y = 0$$

For this system to have a nonzero solution (x, y), we need

$$\det\begin{pmatrix} 2-\alpha & 2 \\ 1 & 3-\alpha \end{pmatrix} = 0,$$

(because otherwise the linear map defined by the matrix is invertible and has trivial kernel) implying $\alpha^2 - 5\alpha + 4 = 0$, i.e. $\alpha = 1$ or 4.

If $\alpha = 1$, the system is equivalent to $x + 2y = 0$.

If $\alpha = 4$, the system is equivalent to $x - y = 0$.

Thus

$$V_1 = \{(x, y) : x + 2y = 0\} = \mathbb{R}(-2, 1)$$
$$V_4 = \{(x, y) : x - y = 0\} = \mathbb{R}(1, 1)$$
$$V_\alpha = 0 \text{ if } \alpha \neq 1, 4.$$

In short 1 and 4 are the two eigenvalues and $(-2, 1)$ and $(1, 1)$, or their nonzero multiples, are the corresponding eigenvectors.

III.3. *Let V be the vector space of real sequences and let* φ *be the linear map from V into V defined by* $\varphi(x_0, x_1, x_2, \ldots) = (x_1, x_2, x_3, \ldots)$. *Find the eigenvalues and eigenvectors of* φ. (4 pts.)

Solution. We need to solve $\alpha(x_0, x_1, x_2, \ldots) = \varphi(x_0, x_1, x_2, \ldots) = (x_1, x_2, x_3, \ldots)$, i.e.

$$x_1 = \alpha x_0$$
$$x_2 = \alpha x_1$$
$$x_3 = \alpha x_2$$
$$x_4 = \alpha x_3$$
$$\ldots\ldots$$

Any α is an eigenvalue. The vector $(1, \alpha, \alpha^2, \alpha^3, \ldots.)$ is an eigenvector for α.

III.4. *Let* $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ *be a rotation* (*around* (0,0) *of course*, *otherwise* φ *is not linear*). *Can* φ *have an eigenvalue*? (10 pts.)

Answer. No, unless it is a rotation of π or 2π radians, in which cases the eigenvalues are -1 and 1 respectively. Why? Well, how can you rotate a vector and still get a scalar multiple of that vector?

III.5. *Assume V is finite dimensional. Show that* α *is an eigenvalue for* φ *if and only if* α *is a root of the polynomial* $\det(\varphi - X\mathrm{Id}_V) = 0$. (*Here I assume that you know that a linear map from a finite dimensional vector space into itself is invertible iff its determinant is nonzero*). *Check this result on Question* 2. *Conclude that a linear map* $\mathbb{R}^3$ *into itself has always an eigenvector*. (10 pts.)

Proof: Let α be an eigenvalue for φ. Then there is a nonzero vector such that $\varphi(v) = \alpha v$. Hence $(\varphi - \alpha\mathrm{Id})(v) = 0$. It follows that $v \in \mathrm{Ker}(\varphi - \alpha\mathrm{Id})$ and $\varphi - \alpha\mathrm{Id}$ is noninvertible. Hence $\det(\varphi - \alpha\mathrm{Id}) = 0$, i.e. α is a root of $\det(\varphi - X\mathrm{Id}) = 0$.

Conversely, assume that α is a root of $\det(\varphi - X\mathrm{Id}) = 0$. Hence $\det(\varphi - \alpha\mathrm{Id}) = 0$ and the linear map $\varphi - \alpha\mathrm{Id}$ is noninvertible. Let v be a nonzero vector in $\ker(\varphi - \alpha\mathrm{Id})$. Hence $(\varphi - \alpha\mathrm{Id})(v) = 0$ and $\varphi(v) = \alpha v$. Since $v \neq 0$, this shows that α is an eigenvalue of φ.

This is exactly how Question 2 was solved.

If $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$, then the polynomial $\det(\varphi - X\mathrm{Id})$ is of degree 3, and must have a solution.

III.6. *Show that if* $(\alpha_i)_i$ *are all distinct scalars and* $0 \neq v_i \in V_{\alpha_i}$, *then the set* $(v_i)_i$ *is a linearly independent set. In other words, show that the subspace spanned by the subspaces* V_α *is a direct sum of them* (10 pts.)

Proof: Assume $\beta_{i_1}, \beta_{i_2}, \ldots, \beta_{i_n}$ are nonzero scalars and that

$$\beta_{i_1} v_{i_1} + \beta_{i_2} v_{i_2} + \ldots + \beta_{i_n} v_{i_n} = 0.$$

Applying φ to this equality we get

$$\beta_{i_1} \alpha_{i_1} v_{i_1} + \beta_{i_2} \alpha_{i_2} v_{i_2} + \ldots + \beta_{i_n} \alpha_{i_n} v_{i_n} = 0.$$

Subtract the second equation from the first equation multiplied by α_{i_1} to get

$$\beta_{i_2} (\alpha_{i_1} - \alpha_{i_2}) v_{i_2} + \ldots + \beta_{i_n} (\alpha_{i_1} - \alpha_{i_n}) v_{i_n} = 0.$$

Now we have $\leq n - 1$ terms. By induction we can conclude that

$$\beta_{i_2} (\alpha_{i_1} - \alpha_{i_2}) = \ldots = \beta_{i_n} (\alpha_{i_1} - \alpha_{i_n}) = 0.$$

Since $\beta_{i_1}, \beta_{i_2}, \ldots, \beta_{i_n}$ are nonzero scalars, we get,

$$\alpha_{i_1} - \alpha_{i_2} = \ldots = \alpha_{i_1} - \alpha_{i_n} = 0,$$

i.e.

$$\alpha_{i_1} = \alpha_{i_2} = \ldots = \alpha_{i_n}.$$

Hence $n = 1$ and the result is trivial in this case.

III.7. *Assume that V is finite dimensional. Show that* $V = \oplus_i V_{\alpha_i}$ *if and only if there is a basis of V in which the matrix of* φ *is diagonal*. (10 pts.)

Proof: Trivial.

Suppose there is a basis of V in which the matrix of φ is diagonal. Let $(v_i)_i$ be this basis. Then $\varphi(v_i) = \alpha_i v_i$ for some scalar α_i. (The α_i is the scalar that appears on the i-th column of the diagonal matrix). It is now easy to check that $V = \oplus_i V_{\alpha_i}$.

Conversely, suppose $V = \oplus_i V_{\alpha_i}$. Choose a basis of each V_{α_i} and collect them together to get a basis of V. The matrix of φ with respect to this basis must be diagonal.

III.8. *Find a basis of* $\mathbb{R}^2$ *in which the matrix of the linear map* φ *in Question* III.2 *is diagonal*. (10 pts.).

Solution. Take $v_1 = (-2, 1)$ and $v_4 = (1, 1)$. Then the matrix of φ with respect to this basis is

$$\begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}.$$