Linear Algebra

Math 231, Midterm 1
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1. Let $V = \{(x, y, z, t, u) \in \mathbb{R}^5 : x + y + z - t + u\}$. Let
$$W = \{(x, y, z, t, u) \in V : 2x - 2y + z - t + u = 0 \text{ and } 2x - 3t = 0\}.$$
We know that V is a vector space over $\mathbb{R}$ and that W is a subspace of V. Find bases of V, W and V/W.

2. Let V be a vector space over a field K. Let $f \in \mathrm{End}_K(V)$ be such that $f \circ f = f$.
2a. Show that $v - f(v) \in \mathrm{Ker}(f)$ for all $v \in V$.
2b. Show that $\mathrm{Ker}(f) \cap \mathrm{Im}(f) = \{0\}$.
2c. Show that $V = \mathrm{Ker}(f) \times \mathrm{Im}(f)$

3. Show that, as vector spaces over $\mathbb{Q}$, $\mathbb{R} \approx \mathbb{R} \times \mathbb{R}$. Conclude that as abelian groups $\mathbb{R} \approx \mathbb{R} \times \mathbb{R}$.

4. For this question, you may assume basic facts of analysis.

Let $V = \{f : [0, 2\pi] \to \mathbb{R} : f \text{ is continuous}\}$. We know (from calculus) that V is a vector space over $\mathbb{R}$. For $f, g \in V$, let $\langle f, g\rangle = \int_0^{2\pi} f(x)g(x)dx$. (If f and g are in V, one can show in analysis that $\langle f, g\rangle$ exists).
4a. Find a subspace W of V such that $\dim(V/W) = 1$.
4b. Show that $\langle\ ,\ \rangle$ is a **bilinear symmetric map** from $V \times V$ into $\mathbb{R}$, i.e. that for all $f, f_1, f_2, g, g_1, g_2 \in V$ and $r \in \mathbb{R}$,
(i) $\langle f_1 + f_2, g\rangle = \langle f_1, g\rangle + \langle f_2, g\rangle$
(ii) $\langle f, g_1 + g_2\rangle = \langle f, g_1\rangle + \langle f, g_2\rangle$
(iii) $\langle rf, g\rangle = r\,\langle f, g\rangle$
(iv) $\langle f, g\rangle = \langle g, f\rangle$
4c. Show that for all $f \in V$, $\langle f, f\rangle \geq 0$. Is it true that if $\langle f, f\rangle = 0$ then $f = 0$.
4d. Show that the set $\{\sin^n x : n \in \mathbb{N}\}$ is a linearly independent subset of V.

5. Let $k \leq K \leq L$ be field extensions. Show that $\dim_k(L)$ is finite if and only if $\dim_k(K)$ and $\dim_K(L)$ are finite. Show that in that case we have $\dim_k(L) = \dim_k(K)\dim_K(L)$. (10 pts.)

6. Let V be a vector space over a field F. Let $(v_i)_{i \in I}$ be a basis of V. For $j \in I$, define a function v_j^* from V into F as follows:

For all $v \in V$, if $v = \sum_{i\in I} \alpha_i v_i$, then $v_j^*(v) = \alpha_j$.

Note that the set of linear maps from V into F form a vector space V^*.

6a. Show that $v_j^* \in V^*$ for all $j \in J$.

6b. Show that the linear maps v_j^* are linearly independent. (10 pts.)

6c. Assume V is finite dimensional. Let $f \in V^*$ and set $f(v_i) = \beta_i \in F$. Show that

$$f = \sum_{i\in I} \beta_i v_i^* .$$

Conclude that the set $(v_i^*)_{i\in I}$ form a basis of V^*.

6d. Assume V is infinite dimensional. Define f: $V \to F$ as follows:

For $v \in V$, if $v = \sum_{i\in I} \alpha_i v_i$, then $f(v) = \sum_{i\in I} \alpha_i$.

Show that $f \in V^*$ but that f is not in the subspace of V^* generated by $(v_i^*)_{i\in I}$.