1. Find all abelian groups of order 1728. How many of them are there?

2. Let U be the subset of $\mathrm{Mat}_3(\mathbb{R})$ consisting of those matrices whose elements below the diagonal are 0, those on the diagonal are nonzero. Show that U is a group under multiplication. Find $[U, U]$. Show that U is nilpotent. Find its class. Generalize your answer as much as you can.

3. An abelian group M has a composition series if and only if M is finite.

4. If G is a finitely generated abelian group, then every surjective endomorphism of G is an automorphism.

5. Let R be a commutative ring with identity, and let D be the subset of R consisiting of the zero divisors in R. Then D contains at least one prime ideal of R.

6. Let R be a commutative ring with identity and C be a subset of R having the following properties

(i) $R \setminus C$ is closed under multiplication.

(ii) For any $c \in C$, we have $Rc \subseteq C$.

Show that C is a union of prime ideals.

7. Let R be a ring with identity, and let $\mathcal{P}$ be a collection of ideals in R.

For each subset A of R, set $\Gamma_A := \{P \in \mathcal{P} : A \not\subseteq P\}$. Put $\tau := \{\Gamma_A \in \mathcal{P} : A \subseteq R\}$. Prove that τ is a topology on $\mathcal{P}$, i.e., show that $\varnothing$, $\mathcal{P}$ belong to τ, that the intersection of any two sets in τ belongs to τ, that the union of any family of sets in τ also belongs to τ.

8. Keep the notation of problem 7. If $\mathcal{P}$ contains all maximal ideals of R, show that $(\mathcal{P}, \tau)$ is compact.

9. Let R be a ring and let

$$\begin{array}{ccccc} A & \xrightarrow{\lambda} & B & \xrightarrow{\mu} & C \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} \\ A' & \xrightarrow[\lambda']{} & B' & \xrightarrow[\mu']{} & C' \end{array}$$

be a commutative diagram of R-modules and R-module homomorphisms with exact rows. Prove that

(i) if α, γ and λ' are monomorphisms, then β is a monomorphism;

(ii) if α, γ and μ are epimorphisms, then β is an epimorphism.

10. Let R be a ring and let

$$\begin{array}{ccccc} A & \xrightarrow{\lambda} & B & \xrightarrow{\mu} & C \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} \\ D & \xrightarrow[\lambda']{} & E & \xrightarrow[\mu']{} & F \end{array}$$

be a commutative diagram of R-modules and R-module homomorphisms with exact rows. Prove that

$$\frac{\mathrm{Im}\,(B \to E) \cap \mathrm{Im}\,(D \to E)}{\mathrm{Im}\,(A \to E)} \cong \frac{\mathrm{Ker}\,(B \to F)}{\mathrm{Ker}\,(B \to C) + \mathrm{Ker}\,(B \to E)}$$

(where, for example, $B \to E$ stands for the map β from B to E).