1. Find all abelian groups of order 1728. How many of them are there?

2. Let $U$ be the subset of $\text{Mat}_3(\mathbb{R})$ consisting of those matrices whose elements below the diagonal are 0, those on the diagonal are nonzero. Show that $U$ is a group under multiplication. Find $[U,U]$. Show that $U$ is nilpotent. Find its class. Generalize your answer as much as you can.

3. An abelian group $M$ has a composition series if and only if $M$ is finite.

4. If $G$ is a finitely generated abelian group, then every surjective endomorphism of $G$ is an automorphism.

5. Let $R$ be a commutative ring with identity, and let $D$ be the subset of $R$ consisting of the zero divisors in $R$. Then $D$ contains at least one prime ideal of $R$.

6. Let $R$ be a commutative ring with identity and $C$ be a subset of $R$ having the following properties

   (i) $R \setminus C$ is closed under multiplication.
   (ii) For any $c \in C$, we have $Rc \subseteq C$.

   Show that $C$ is a union of prime ideals.

7. Let $R$ be a ring with identity, and let $\mathcal{P}$ be a collection of ideals in $R$.

   For each subset $A$ of $R$, set $\Gamma_A := \{ P \in \mathcal{P} : A \not\subseteq P \}$. Put $\tau := \{ \Gamma_A \in \mathcal{P} : A \subseteq R \}$. Prove that $\tau$ is a topology on $\mathcal{P}$, i.e., show that $\emptyset, \mathcal{P}$ belong to $\tau$, that the intersection of any two sets in $\tau$ belongs to $\tau$, that the union of any family of sets in $\tau$ also belongs to $\tau$.

8. Keep the notation of problem 7. If $\mathcal{P}$ contains all maximal ideals of $R$, show that $(\mathcal{P}, \tau)$ is compact.

9. Let $R$ be a ring and let

   
   $\hfill A \xrightarrow{\lambda} B \xrightarrow{\mu} C \\
   \alpha \downarrow \hfill \beta \hfill \gamma \downarrow \\
   A' \xrightarrow{\lambda'} B' \xrightarrow{\mu'} C'$

   be a commutative diagram of $R$-modules and $R$-module homomorphisms with exact rows. Prove that

   (i) if $\alpha, \gamma$ and $\lambda'$ are monomorphisms, then $\beta$ is a monomorphism;
   (ii) if $\alpha, \gamma$ and $\mu$ are epimorphisms, then $\beta$ is an epimorphism.
10. Let $R$ be a ring and let

\[
\begin{array}{ccc}
A & \xrightarrow{\lambda} & B \\
\downarrow{\alpha} & & \downarrow{\beta} \\
D & \xrightarrow{\lambda'} & E \\
\end{array}
\quad
\begin{array}{ccc}
B & \xrightarrow{\mu} & C \\
\downarrow{\beta} & & \downarrow{\gamma} \\
E & \xrightarrow{\mu'} & F \\
\end{array}
\]

be a commutative diagram of $R$-modules and $R$-module homomorphisms with exact rows. Prove that

\[
\frac{\text{Im} (B \to E) \cap \text{Im} (D \to E)}{\text{Im} (A \to E)} \cong \frac{\text{Ker} (B \to F)}{\text{Ker} (B \to C) + \text{Ker} (B \to E)}
\]

(where, for example, $B \to E$ stands for the map $\beta$ from $B$ to $E$).