Linear Algebra

Resit (2)
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Justify all your answers. An answer without justification will receive a zero grade. All vector spaces may be considered as vector spaces over the field $\mathbb{R}$ of real numbers.

1. Give an example of a system of three linear equations with three unknowns
1a. with only one solution. (2 pts.)
1b. with infinitely many solutions. (2 pts.)
1c. with no solutions. (2 pts.)

2. Prove that there cannot be a fourth case in Question 1. (4 pts.)

3. Draw the plane
3a. $\{(2x, 3y, 4x + 6y) : x, y \in \mathbb{R}\}$
3b. $\{(x, y, z) \in \mathbb{R}^3 : 2x + y - z = 0\}$.
in 3-dimensional space (represented in your two dimensional paper!) (6 pts.)

4. Is the set
4b. $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 - z^2 = 0\}$
4c. $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^4 = 0\}$
a vector space? (6 pts.)

5. What is the equation of the plane passing through the points (1, 1, 1), (2, 1, 0) and (1, 3, −1)? (5 pts.)

6. Let W be a subspace of $\mathbb{R}^6$. Show that
$$W_1 = \{(x_1, ..., x_6) : x_1 + ... + x_6 \in W\}$$
is a subspace of $\mathbb{R}^6$. What is the dimension of W_1? (10 pts.)

7. Let V be a vector space and $f : V \to V$ be a linear map.
7a. Show that $\text{Ker} f^2 \subseteq \text{Ker} f$. (3 pts.)
7b. Find an example where $\text{Ker} f^2 \neq \text{Ker} f$. (4 pts.)
7c. Find an example where $\text{Ker} f^2 = \text{Ker} f \neq 0$. (4 pts.)

8. Find the vector space generated by
8a. the set $\{(x, y, z) : x^2 + y^2 = 1\}$ (4 pts.)
8b. the vectors (3, 4, 7) and (1, 4, 5). (3 pts.)
8c. the vector (3, 4, 7). (3 pts.)

9a. Show that the vectors (1, 2, 1), (1, 1, 1) and (2, 1, 1) form a basis of $\mathbb{R}^3$. (4 pts.)

9b. Write the vector (0, 2, 1) as a linear combination of these vectors. (3 pts.)

9c. Write the vector (0, 0, 0) as a linear combination of these vectors. (2 pts.)

9d. What is the matrix of the linear map $(x, y, z) \to (x - y, z - t, t + x)$ with respect to this basis? (4 pts.)

9e. What is the kernel of the above map? (4 pts.)

10. Let V be a vector space. Let $f : V \times V \to V$ be defined by $f(v, w) = v + w$.

10a. Show that the map f is linear. (3 pts.)

10b. What is its kernel? (4 pts.)

10c. What is its image? (4 pts.)

10d. Find a linear map $g : V \to V \times V$ such that $f \circ g = \mathrm{Id}_V$. (7 pts.)

10e. Show that there is no a linear map $g : V \to V \times V$ such that $g \circ f = \mathrm{Id}_{V\times V}$. (7 pts.)