Linear Algebra II

Resit
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1. Let V be the vector space of functions from $\mathbb{R}$ into $\mathbb{R}$ which has $\{\sin\theta, \cos\theta\}$ as a basis. Let $D : V \to V$ be the differentiable operator defined by $D(f) = f'$. Show that D is linear. What is its matrix? Find the minimal polynomial of D. Find all iterates of D.

2. Let $\ell_2 = \{(a_n)_{n \in \mathbf{N}} : a_n \in \mathbb{R}$ and $\sum_{n\in \mathrm{N}} {a_n}^2$ converges$\}$. Show that ℓ_2 is a vector space over $\mathbb{R}$. Show that its dimension is uncountable.

3. Let A and B be two square matrices (over a field) of the same dimension.
3a. Show that AB is not invertible iff either A or B is noninvertible.
3b. Show that 0 is an eigenvalue of AB iff 0 is an eigenvalue of BA.
3c. Show that AB and BA have the same eigenvalues.
3d. Show that $\mathrm{tr}(AB - BA) = 0$.

4a. Let $K \le L$ be a field extension. The field L can be considered as a vector space over K in a natural way. Assume $\dim_K(L) = n < \infty$. Show that the field L is isomorphic to a subring of $M_{n\times n}(K)$.

4b. Assume $K = \mathbb{R}$ and $L = \mathbb{C}$. Find explicitely the subring of $M_{2\times 2}(\mathbb{R})$ which is isomorphic to $\mathbb{C}$.

5. Let V be a vector space of dimension n. Let $T \in \mathrm{End}(V)$ be such that $T^k = 0$ for some natural number k. Assume k is minimum such. Show that $k \le n$.

6. Let V be a vector space over a field K. Let $(v_i)_{i \in I}$ be a basis of V. For $j \in I$, define a function $v_j{}^*$ from V into K as follows:

For all $v \in V$, if $v = \sum_{i\in I} \alpha_i v_i$, then $v_j{}^*(v) = \alpha_j$.

Let $V^* = \mathrm{End}_K(V, K)$.
6a. Show that $v_j{}^* \in V^*$ for all $j \in J$.
6b. Show that the linear maps $v_j{}^*$ are linearly independent.
6c. Show that V is finite dimensional iff $(v_i{}^*)_{i\in I}$ form a basis of V^*.

7. Let V be a vector space over a field K. Let $V^{**} = \mathrm{End}_K(V^*, K)$. For $v \in V$ and $f \in V^{**}$, define $v^{**} : V^* \to K$ by $v^{**}(f) = f(v)$. (Note that, unlike in V^{**}, the ** in v^{**} is just one symbol and has nothing to do with the previous *'s).
7a. Show that $v^{**} \in V^{**}$.
7b. Show that the map $v \mapsto v^{**}$ is a homomorphism of vector spaces from V into V^{**}. Call it **
7c. Show that ** is one-to-one.
7d. Show that ** is an isomorphism iff $\dim_K(V) < \infty$.

8. Let V be a vector space over $\mathbf{K} = \mathbb{R}$ or $\mathbb{C}$. A map $\langle\ ,\ \rangle : V \times V \to \mathbf{K}$ is called an inner product in V if

a. $\langle\ ,\ \rangle$ is linear in the first component.

b. $\langle u, v\rangle = \overline{\langle v, u\rangle}$ all $u, v \in V$.

c. $\langle v, v\rangle \geq 0$ all $v \in V$ and it is zero iff $v = 0$.

8a. Show that $\langle A, B\rangle = \mathrm{tr}(B^{t}A)$ defines an inner product on the space of $n \times m$ matrices over $\mathbb{R}$.

8b. Show that, on ℓ_2 the "product" $\langle (a_n)_n, \langle (b_n)_n\rangle = \sum a_n b_n$ makes sense and that it is an inner product.

Let V be a vector space with an inner product $\langle\ ,\ \rangle$. We let $|v| = \sqrt{\langle v, v\rangle}$.

8c. Show that the formula $d(u, v) = |u - v|$ defines a distance on V.

8d. (Cauchy-Schwartz) Show that $|\langle u, v\rangle| \leq |u|\,|v|$ for all u, v in V.

8e*. For $W \leq V$, define $W^{\perp} = \{v \in V : \langle v, W\rangle = 0\}$. Show that $V = W \oplus W^{\perp}$.