1. Let $V$ be the vector space of functions from $\mathbb{R}$ into $\mathbb{R}$ which has \{sin $\theta$, cos $\theta$\} as a basis. Let $D : V \to V$ be the differentiable operator defined by $D(f) = f'$. Show that $D$ is linear. What is its matrix? Find the minimal polynomial of $D$. Find all iterates of $D$.

2. Let $\ell_2 = \{(a_n)_{n \in \mathbb{N}} : a_n \in \mathbb{R}$ and $\sum_{n \in \mathbb{N}} a_n^2$ converges\}. Show that $\ell_2$ is a vector space over $\mathbb{R}$. Show that its dimension is uncountable.

3. Let $A$ and $B$ be two square matrices (over a field) of the same dimension.

3a. Show that $AB$ is not invertible iff either $A$ or $B$ is noninvertible.

3b. Show that 0 is an eigenvalue of $AB$ iff 0 is an eigenvalue of $BA$.

3c. Show that $AB$ and $BA$ have the same eigenvalues.

3d. Show that $\text{tr}(AB - BA) = 0$.

4a. Let $K \leq L$ be a field extension. The field $L$ can be considered as a vector space over $K$ in a natural way. Assume $\dim_K(L) = n < \infty$. Show that the field $L$ is isomorphic to a subring of $M_{n \times n}(K)$.

4b. Assume $K = \mathbb{R}$ and $L = \mathbb{C}$. Find explicitly the subring of $M_{2 \times 2}(\mathbb{R})$ which is isomorphic to $\mathbb{C}$.

5. Let $V$ be a vector space of dimension $n$. Let $T \in \text{End}(V)$ be such that $T^k = 0$ for some natural number $k$. Assume $k$ is minimum such. Show that $k \leq n$.

6. Let $V$ be a vector space over a field $K$. Let $(v_i)_{i \in I}$ be a basis of $V$. For $j \in I$, define a function $v_j^*$ from $V$ into $K$ as follows:

For all $v \in V$, if $v = \sum_{i \in I} \alpha_i v_i$, then $v_j^*(v) = \alpha_j$.

Let $V^* = \text{End}_K(V, K)$.

6a. Show that $v_j^* \in V^*$ for all $j \in J$.

6b. Show that the linear maps $v_j^*$ are linearly independent.

6c. Show that $V$ is finite dimensional iff $(v_i^*)_{i \in I}$ form a basis of $V^*$.

7. Let $V$ be a vector space over a field $K$. Let $V'' = \text{End}_K(V', K)$. For $v \in V$ and $f \in V''$, define $v^{**} : V' \to K$ by $v^{**}(f) = f(v)$. (Note that, unlike in $V''$, the ** in $v^{**}$ is just one symbol and has nothing to do with the previous *’s).

7a. Show that $v^{**} \in V^{**}$.

7b. Show that the map $v \mapsto v^{**}$ is a homomorphism of vector spaces from $V$ into $V^{**}$. Call it $**$

7c. Show that $**$ is one-to-one.

7d. Show that $**$ is an isomorphism iff $\dim_K(V) < \infty$. 

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8. Let $V$ be a vector space over $K = \mathbb{R}$ or $\mathbb{C}$. A map $\langle \cdot , \cdot \rangle : V \times V \to K$ is called an inner product in $V$ if

a. $\langle \cdot , \cdot \rangle$ is linear in the first component.

b. $\langle u, v \rangle = \langle v, u \rangle$ all $u, v \in V$.

c. $\langle v, v \rangle \geq 0$ all $v \in V$ and it is zero iff $v = 0$.

8a. Show that $\langle A, B \rangle = \text{tr}(B^tA)$ defines an inner product on the space of $n \times m$ matrices over $\mathbb{R}$.

8b. Show that, on $\ell_2$ the “product” $\langle (a_n)_n, (b_n)_n \rangle = \sum a_n b_n$ makes sense and that it is an inner product.

Let $V$ be a vector space with an inner product $\langle \cdot , \cdot \rangle$. We let $|v| = \sqrt{\langle v, v \rangle}$.

8c. Show that the formula $d(u, v) = |u - v|$ defines a distance on $V$.

8d. (Cauchy-Schwartz) Show that $|\langle u, v \rangle| \leq |u| |v|$ for all $u, v$ in $V$.

8e*. For $W \leq V$, define $W^\perp = \{ v \in V : \langle v, W \rangle = 0 \}$. Show that $V = W \oplus W^\perp$. 