Linear Algebra

Resit

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Justify all your answers. An answer without justification will receive a zero grade. All vector spaces may be considered as vector spaces over the field $\mathbb{R}$ of real numbers.

1a. Is the set $\{(x, y, z) \in \mathbb{R}^3 : x + y + z = 0\}$ a vector space?

1b. Is the set $\{(x, y, z) \in \mathbb{R}^3 : x + y + z \in \mathbb{Z}\}$ a vector space?

1c. Is the set $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 0\}$ a vector space?

1d. Is the set $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 - z^2 = 0\}$ a vector space?

2a. What is the subspace of $\mathbb{R}$ generated[1] by $\sqrt{2}$?

2b. What is the subspace of $\mathbb{R}^2$ generated[2] by $(1, \pi)$ and $(\sqrt{2}, -3)$?

2c. For what values of a and b does (b, a) belong to the subspace of $\mathbb{R}^2$ generated by (a, b)?

3. Find

$$\begin{pmatrix} 1 & 1 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}^{-1}$$

4. Consider the set A of $(x, y, z) \in \mathbb{R}^3$ such that

$$A = \{(x, y, z) \in \mathbb{R}^3 : \det\begin{pmatrix} 3 & 2 & 0 \\ x & y & z \\ -1 & -1 & 1 \end{pmatrix} = 0\}.$$

Show that A is a subspace of $\mathbb{R}^2$. What is its dimension? Find a basis of A.

5. Show that, for every fixed t, the set A_t of $(x, y, z) \in \mathbb{R}^3$ such that

$$\det\begin{pmatrix} 3 & 2 & 0 \\ x & y & z \\ t & -1 & t \end{pmatrix} = 0.$$

is a subspace. What is the dimension of A_t? (Note that the dimension may depend on t). Find a basis of A_t.

6. Let $V = \{(x, y, z, t, u) \in \mathbb{R}^5 : x - y + 2z - t + u\}$. Let

[1] Subspace generated by an element is the smallest subspace containing this element.

[2] Subspace generated by two elements is the smallest subspace containing these elements.

$$W = \{(x, y, z, t, u) \in V : x - 2y + z - t + u = 0 \text{ and } 2x - 3t = 0\}.$$

We know that V is a vector space over $\mathbb{R}$ and that W is a subspace of V. Find a basis of V, W and V/W.

7. Let $V = \{f : \mathbb{R} \to \mathbb{R} : f \text{ is differentiable}\}$. We know (from calculus) that V is a vector space over $\mathbb{R}$.

7a. Show that the set $(x^n)_{n \in \mathbb{N}}$ is a linearly independent subset of V.

7b. Show that the set $(\sin^n x)_{n \in \mathbb{N}}$ is a linearly independent subset of V.

8. Let V be a vector space. Let $f : V \to V$ be a linear map such that $f \circ f = f$.

8a. Show that $v - f(v) \in \mathrm{Ker}(f)$ for all $v \in V$.

8b. Show that $\mathrm{Ker}(f) \cap \mathrm{Im}(f) = \{0\}$.

8c. Show that $V = \mathrm{Ker}(f) \oplus \mathrm{Im}(f)$.

8d. Show that f is the identity map on $\mathrm{Im}(f)$ and that it is the zero map on $\mathrm{Ker}(f)$.