Math 232

Midterm
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Throughout K denotes a field.

1. Compute the determinent

$$\begin{bmatrix} a_0 & a_1 & a_2 \cdots & a_{n-1} & a_n \\ -1 & x & 0 \cdots & 0 & 0 \\ 0 & -1 & x \cdots & 0 & 0 \\ \vdots & \vdots & \vdots \cdots & \vdots & 0 \\ 0 & 0 & 0 \cdots & x & 0 \\ 0 & 0 & 0 \cdots & -1 & x \end{bmatrix}$$

2. Let $f : K^3 \to K^4$ be defined by $f(x, y, z) = (x - y, 0, 2x - 2y, x - y - 2z)$.

2a. Find a basis of $\mathrm{Im}(f)$.

2b. Find a basis of $\mathrm{Ker}(f)$.

3. Let $W = \{(x - y, x - y + z, z, 0, 2z) : x, y, z \in K\}$. Find a basis of the quotient space K^5/W.

4. What is the dimension of the intersection of two distinct subspaces of dimension $n - 1$ of a vector space of dimension n?

5. Let $f(X) = a_o + a_1X + ... + a_nX^n \in K[X]$ where $a_n \neq 0$ and let $V = K[X]/(f)$. We consider V as a vector space over K.

5a. Find a basis of V.

5b. Multiplication by X gives rise to a linear map, say φ, on V. Write the matrix of this map with respect to the basis you found above.

5c. Compute the determinent of the matrix above. When is it 0?

5d. What is the kernel of φ?

6. If $A = (a_{ij})_{ij}$ is an $n \times n$ square matrix K, the trace of A, $\mathrm{tr}(A)$, is defined to be $\sum_{i=1}^{n} a_{ii}$.

6a. Show that tr is a linear map from the vector space of $n \times n$ matrices over K into K.

6b. What is the dimension of the kernel of tr?

7a. Let $f : K^n \to K$ be a linear map. Show that there are $a_1, a_2,..., a_n \in K$ such that $f(x_1, x_2,..., x_n) = a_1x_1 + a_2x_2 + ... + a_nx_n$ for all $(x_1, x_2,..., x_n) \in K^n$.

7b. Conclude that every subspace of dimension $n - 1$ of K^n is of the form

$$\{(x_1, x_2,..., x_n) \in K^n : a_1x_1 + a_2x_2 + ... + a_nx_n = 0\}$$

for some $a_1, a_2,..., a_n \in K$ not all 0.

8. Call the 1-dimensional subspaces of K^3 *points*, and the set of 2-dimensional subspaces of K. We will say that a point P is on the line ℓ if $P \subseteq \ell$. Show that

8a. Through any two points a unique line passes.

8b. Two distinct lines intersect in a unique point.

8c. If $K = \mathbf{F}_q$, how many lines and points does this geometry have?

8d. Show that, whatever the field is, there is a (quite natural) bijection between the set of lines and points of this geometry. (Hint: see #7b).

8e. Draw the points and the lines of this geometry if $K = \mathbf{F}_2$.