Math 232  
(Linear Algebra)  
Midterm 2 for CS majors  
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I. Are the following sets vector spaces over \( \mathbb{R} \)?

a. \( A = \{ f : \mathbb{R} \to \mathbb{R} : f(x) \geq 0 \text{ for all } x \in \mathbb{R} \} \)

b. \( B = \{ f : \mathbb{R} \to \mathbb{R} : f(x) = 0 \text{ for all } x \geq 0 \} \)

c. \( C = \{ (x, y, z) \in \mathbb{R}^3 : x + y + z \in \mathbb{Z} \} \)

d. \( D = \{ (x, x^2) \in \mathbb{R}^2 : x \in \mathbb{R} \} \)

Justify your answers.

II. Find a basis of the following vector space:
\( \{ (2z, x - y, x - y, z + t, z - t, u + x - y) : (x, y, z, t, u) \in \mathbb{R}^6 : x + y + z + t + u = 0 \} \).

III. Find the general form of a vector of the subspace of \( \mathbb{R}^4 \) generated by the vectors
\( v_1 = (1, 2, 3, 1) \)
\( v_2 = (1, 2, 3, 2) \)
\( v_3 = (1, 2, 3, -1) \)

IV. Let \( f : \mathbb{R}^3 \to \mathbb{R}^4 \) be defined by \( f(x, y, z) = (x - y, 0, 2x - 2y, x - y - 2z) \).

IV.1. Show that \( f \) is a linear map.

IV.2. Find a basis of \( \text{Im}(f) \).

IV.3. Find a basis of \( \text{Ker}(f) \).

V. Let \( W = \{ (x - y, x - y + z, z, 0, 2z) : x, y, z \in \mathbb{R} \} \). \( W \) is a subspace of \( \mathbb{R}^5 \). Find a basis of the quotient space \( \mathbb{R}^5/W \).

VI. Let \( f : V \to W \) be a linear map between two vector spaces \( V \) and \( W \).

VI.1. Show that if \( v_1, \ldots, v_n \in V \) are such that \( f(v_1), \ldots, f(v_n) \) are linearly independent, then \( v_1, \ldots, v_n \) are also linearly independent.

VI.2. Conclude that \( \dim V \geq \dim f(V) \).