Math 232
(Linear Algebra)

Midterm 1 for Math majors
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I. Let K be a field, A a set and Π a set of subsets of A. Let V be the set of functions from A into K. V is a vector space over K with the usual operations (addition and scalar multiplication). Let $V(\Pi)$ be the set of elements of V that vanish on a set that belongs to Π. Thus,

$$V(\Pi) = \{ f : A \to K : \text{there is } X \in \Pi \text{ such that } f = 0 \text{ on } X\}.$$

I.1. Find the necessary and sufficent condition for $V(\Pi)$ to be a subspace of V.

I.2. Is every subspace of V of the form $V(\Pi)$ for some set Π of subsets of A?

II. Let V and W be two vector spaces of dimension n and m over the same field K. Show that $V \times W$ is a vector space of dimension $n + m$.

III. Let V be a vector space and A and B be two subspaces of V. Show that $\dim(A + B) = \dim(A) + \dim(B) - \dim(A \cap B)$

IV. Let V and W be two vector spaces over the same field.

IV.1. Show that the set of linear maps $\text{Hom}_K(V, W)$ is a vector space over the same field.

Assume from now on that $\dim(V) = n$ and $\dim(W) = m$. Let $v_1, ..., v_n$ and $w_1, ..., w_m$ be bases of V and W respectively. For $i = 1, ..., n$ and $j = 1, ..., m$ define $f_{ij} : V \to W$ by the rule $f_{ij}(\alpha_1 v_1 + ... + \alpha_n v_n) = \alpha_i w_j$.

IV.2. Show that $f_{ij} \in \text{Hom}_K(V, W)$.

IV.3. Show that the set $\{f_{ij} : i = 1, ..., n ;\ j = 1, ..., m\}$ is linearly independent.

IV.4. Show that the set $\{f_{ij} : i = 1, ..., n ;\ j = 1, ..., m\}$ is a basis of $\text{Hom}_K(V, W)$.

IV.5. Show that $\dim(\text{Hom}_K(V, W)) = nm$.

V. Show that $\mathbf{F}_q^{\ n}$ has exactly $(q^n - 1)(q^n - q) \ldots (q^n - q^{n-1})$ different basis. (**Hint:** Choose the vectors of a basis one by one! The first one cannot be the zero vector...)

VI. Let V be a vector space over a field K. Show that the set $GL(V)$ of K-vector space automorphisms of V form a group under the composition.

VII. Let $V = \mathbf{F}_q^{\ n}$. Show that $GL(V)$ has $(q^n - 1)(q^n - q) \ldots (q^n - q^{n-1})$ elements. (**Hint:** $GL(V)$ is defined in question IV. Look at question V).