Algebra
(Math 211)
First Midterm

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Throughout G stands for a group.

1. Let $H, K \leq G$. Show that $\{HxK : x \in G\}$ is a partition of G. (3 pts.)

   Proof: The relation $x \equiv y$ defined by "$HxK = HyK$" is certainly reflexive and symmetric. Let us prove the transitivity. It is clear that $HxK = HyK$ if and only if $x \in HyK$. Thus if $x \in HyK$ and $y \in HzK$, then $x \in HHzKK \subseteq HzK$.

2. Let $H \leq G$. Show that there is a natural one to one correspondence between the left coset space of H in G and the right coset space of H in G. (3 pts.)

   Proof: Consider the map $xH \mapsto Hx^{-1}$. This is well defined and one to one because $xH = yH$ if and only if $y^{-1}x \in H$ if and only if $y^{-1} \in Hx^{-1}$ if and only if $Hy^{-1} = Hx^{-1}$. It is also onto.

3. Let $H, K \leq G$. Show that $xH \cap yK$ is either empty or of the form $z(H \cap K)$ for some $z \in G$. (5 pts.)

   Proof: Assume $xH \cap yK \neq \emptyset$. Let $z \in xH \cap yK$. Then $xH = zH$ and $yK = zK$. So $xH \cap yK = zH \cap zK = z(H \cap K)$.

4. a) Show that the intersection of two subgroups of finite index is finite. (5 pts.)

   b) If $[G : H] = n$ and $[G : K] = m$, what can you say about $[G : H \cap K]$? (7 pts.)

   Proof: (a) Let H and K be two subgroups of index n and m of a group G. Then for any $x \in G$, $x(H \cap K) = xH \cap xK$ and there are at most n choices for xH and m choices for xK. Hence $[G : H \cap K] \leq nm$.

   (b) If $C \leq B \leq A$ and if the indices are finite then $[A : C] = [A : B][B : C]$ because cosets of C partition B and cosets of B partition A, i.e. if $B = \sqcup_{i=1}^{r} b_i C$ and $A = \sqcup_{j=1}^{s} a_j B$, then $A = \sqcup_{i=1}^{r} \sqcup_{j=1}^{s} b_i a_j C$.

Thus $[G : K \cap H] = [G : H][H : H \cap K] = [G : K][K : H \cap K]$. It follows that n and m both divide $[G : K \cap H]$, hence $\mathrm{lcm}(n, m)$ divides $[G : K \cap H]$. Further in part (a) we have seen that $[G : K \cap H] \leq mn$.

5. Let G be a group and $H \leq G$ a subgroup of index n. Let $X = G/H$ be the left coset space. For $g \in G$, define $\tilde{g} : G/H \longrightarrow G/H$ by $\tilde{g}(xH) = gxH$ for $x \in G$.

   a) Show that $\tilde{g} \in \mathrm{Sym}(X)$. (2 pts.)

   b) Show that $\tilde{} : G \longrightarrow \mathrm{Sym}(X)$ is a homomorphism of groups. (3 pts.)

   c) Show that $\mathrm{Ker}(\tilde{})$ is the largest normal subgroup of G contained in H. (5 pts.)

   d) Show that $[G : \mathrm{Ker}(\tilde{})]$ divides $n!$. (5 pts.)

   e) Conclude that there is an $m \in \mathbb{N} \setminus \{0\}$ such that for every $g \in G$, $g^m \in H$. (3 pts.)

   f) Conclude that a divisible group[1] cannot have a proper subgroup of finite index. (7 pts.)

   Proof: (a) $\tilde{g}$ is one to one because if $\tilde{g}(xH) = \tilde{g}(x_1H)$ then $gxH = gx_1H$, and so $xH = x_1H$. $\tilde{g}$ is onto because if $xH \in G/H$, then $\tilde{g}(g^{-1}xH) = xH$.

   (b) Let g, $h \in G$ be any two elements. Since $(\tilde{g} \circ \tilde{h})(xH) = \tilde{g}(\tilde{h})(xH)) = \tilde{g}(hxH) = ghxH = \widetilde{gh}(xH)$ for all $xH \in G/H$, $\tilde{g} \circ \tilde{h}\widetilde{gh}$. Hence $\tilde{}$ is a group homomorphism.

   (c) $\mathrm{Ker}(\tilde{})$ is certainly a normal subgroup of G. Also $\mathrm{Ker}(\tilde{}) = \{g \in G : \tilde{g} = \mathrm{Id}\} = \{g \in G : gxH = xH \text{ for all } x \in G\} = \{g \in G : x^{-1}gx \in H \text{ for all } x \in G\} = \{g \in G : g \in xHx^{-1} \text{ for all } x \in G\} = \cap_{x \in G} H^x$. It is now clear that $\mathrm{Ker}(\tilde{})$ is the largest normal subgroup of G contained in H.

   (d) By above $G/\mathrm{Ker}(\tilde{})$ embeds in $\mathrm{Sym}(G/H) \simeq \mathrm{Sym}(n)$.

   (e) Take $m = n!$.

   (f) Let G be a divisible group and $H \leq G$ a subgroup of index n. Let $g \in G$. Let $h \in G$ be such that $g = h^{n!}$. By the above, $g = h^{n!} \in H$. So $G = H$.

6. Recall that $Z(G) = \{z \in G : zg = gz\}$.

   a) Show that $Z(G) \lhd G$. (3 pts.)

   b) Assume that $G/Z(G)$ is cyclic. Show that G is abelian. (7 pts.)

   Proof: (a) If $z, z_1 \in Z(G)$, then for all $g \in G$, $(zz_1)g = z(z_1g) = z(gz_1) = (zg)z_1 = (gz)z_1 = g(zz_1$, so that $zz_1 \in Z(G)$. Thus $Z(G)$ is closed under multiplication. Clearly $1 \in Z(G)$. Finally, if $z \in Z(G)$, since for all $g \in G$, $gz = zg$, multiplying by z^{-1} from left and right, we see that $gz^{-1} = z^{-1}g$, i.e. $z^{-1} \in Z(G)$. Thus $Z(G)$ is a subgroup.

[1] A group G is called divisible if for any $g \in G$ and any integer $n \geq n$ there is an $h \in G$ such that $g = h^n$.

If $z \in Z(G)$ and $g \in G$, then $g^{-1}zg = z$, so that $g^{-1}Z(G)g \subseteq Z(G)$. This means exactly that $Z(G)$ is a normal subgroup of G.

7. Let G' be the subgroup generated by $\{xyx^{-1}y^{-1} : x, y \in G\}$.

   a) Show that $G' \lhd G$. (5 pts.)

   b) Show that G/G' is abelian. (5 pts.)

   c) Let $H \lhd G$. Show that if G/H is abelian then $G' \leq H$. (5 pts.)

   d) Show that G' is the smallest normal subgroup H of G such that G/H is abelian. (3 pts.)

   e) Let $H = \langle g^2 : g \in G \rangle$. Show that $H \leq G'$. (5 pts.)

   Proof: (a) For $x, y, g \in G$, $g^{-1}(xy)g = (g^{-1}xg)(g^{-1}yg)$ and so $g^{-1}(xyx^{-1}y^{-1})g = (g^{-1}xg)(g^{-1}yg)(g^{-1}xg)^{-1}(g^{-1}yg)^{-1}$. Hence $g^{-1}\langle xyx^{-1}y^{-1} : x, y \in G\rangle g \leq \langle xyx^{-1}y^{-1} : x, y \in G\rangle$, i.e. $G' := \langle xyx^{-1}y^{-1} : x, y \in G\rangle$ is a normal subgroup of G.

   (b) For any $\overline{x}, \overline{y} \in G$, $\overline{x}^{-1}\overline{y}^{-1}\overline{xy} = \overline{x^{-1}y^{-1}xy} = \overline{1}$ because $x^{-1}y^{-1}xy \in G'$.

   (c) For any $x, y \in G$, $\overline{1} = \overline{x}^{-1}\overline{y}^{-1}\overline{xy} = \overline{x^{-1}y^{-1}xy}$, i.e. $x^{-1}y^{-1}xy \in H$. It follows that $G' \leq H$.

   (d) Follows directly from part (c)

   (e) We first claim that if G is a group in which every element has order 2, then G is abelian. Indeed, for any $g, h \in G$, $ghgh = (gh)^2 = 1$, so that $gh = h^{-1}g^{-1} = hg$.

   Now we prove (e). Clearly, for any $\overline{g} \in G/H$, $\overline{g}^2 = \overline{1}$. Such a group must be abelian. Thus $G' \leq H$ by part (c).

8. Let X be a set. Let Γ be the set of subsets of X with two elements. On Γ define the relation $\alpha R \beta$ if and only if $\alpha \cap \beta = \emptyset$. Then Γ becomes a graph with this relation.

   a) Calculate $\mathrm{Aut}(\Gamma)$ when $|X| = 4$. (5 pts.)

   b) Draw the graph Γ when $X = \{1, 2, 3, 4, 5\}$. (3 pts.)

   c) Show that $\mathrm{Sym}(5)$ imbeds in $\mathrm{Aut}(\Gamma)$ naturally. (5 pts.)

   d) Show that $\mathrm{Aut}(\Gamma) \simeq \mathrm{Sym}(5)$. (7 pts.)

   Answer: (a) The graph Γ is just six vertices joined two by two. A group isomorphic to $(\mathbb{Z}/2\mathbb{Z})^3$ preserves the edges. And $\mathrm{Sym}(3)$ permutes the edges. Thus the group has $8 \times 3! = 48$ elements.

   More formally, one can prove this as follows. Let the points be $\{1, 2, 3, 4, 5, 6\}$ and the edges be $v_1 = (1, 4)$, $v_2 = (2, 5)$ and $v_3 = (3, 6)$. We can embed

Sym(3) in $\mathrm{Aut}(\Gamma) \leq \mathrm{Sym}(6)$ via

$$\begin{array}{lcl} \mathrm{Id}_3 & \mapsto & \mathrm{Id}_6 \\ (12) & \mapsto & (12)(45) \\ (13) & \mapsto & (13)(46) \\ (23) & \mapsto & (23)(56) \\ (123) & \mapsto & (123)(456) \\ (132) & \mapsto & (132)(465) \end{array}$$

For any $\phi \in \mathrm{Aut}(\Gamma)$ there is an element α in the image of Sym(3) such that $\alpha^{-1}\phi$ preserves the three edges $v_1 = (1,4)$, $v_2 = (2,5)$ and $v_3 = (3,6)$. Thus $\alpha^{-1}\phi \in \mathrm{Sym}\{1,4\} \times \mathrm{Sym}\{2,5\} \times \mathrm{Sym}\{3,6\} \simeq (\mathbb{Z}/2\mathbb{Z})^3$. It follows that $\mathrm{Aut}(\Gamma) \simeq (\mathbb{Z}/2\mathbb{Z})^3 \rtimes \mathrm{Sym}(3)$ (to be explained next year).

(b) There are ten points. Draw two pentagons one inside the other. Label the outside points as $\{1,2\}$, $\{3,4\}$, $\{5,1\}$, $\{2,3\}$, $\{4,5\}$. Complete the graph.

(c and d) Clearly any element of $\sigma \in \mathrm{Sym}(5)$ gives rise to an automorphism $\tilde{\sigma}$ of Γ via $\tilde{\sigma}\{a,b\} = \{\sigma(a), \sigma(b)\}$. The fact that this map preserves the incidence relation is clear. This map is one to one because if $\tilde{\sigma} = \tilde{\tau}$, then for all distinct a, b, c, we have $\{\sigma(b)\} = \{\sigma(a), \sigma(b)\} \cap \{\sigma(b), \sigma(c)\} = \tilde{\sigma}\{a,b\} \cap \tilde{\sigma}\{b,c\} = \tilde{\tau}\{a,b\} \cap \tilde{\tau}\{b,c\} = \{\tau(a), \tau(b)\} \cap \{\tau(b), \tau(c)\} = \{\tau(b)\}$ and hence $\sigma(b) = \tau(b)$.

Let $\phi \in \mathrm{Aut}(\Gamma)$. We will compose ϕ by elements of Sym(5) to obtain the identity map. There is an $\sigma \in \mathrm{Sym}(5)$ such that $\phi\{1,2\} = \tilde{\sigma}\{1,2\}$ and $\phi\{3,4\} = \tilde{\sigma}\{3,4\}$. Thus, replacing ϕ by $\sigma^{-1}\phi$, we may assume that ϕ fixes the vertices $\{1,2\}$ and $\{3,4\}$. Now ϕ must preserve or exchange the vertices $\{3,5\}$ and $\{4,5\}$. By applying the element (34) of Sym(5) we may assume that these two vertices are fixed as well. Now ϕ must preserve or exchange the vertices $\{1,3\}$ and $\{2,3\}$. By applying the element (12) of Sym(5) we may assume that these two vertices are fixed as well. Now all the vertices must be fixed.