Linear Algebra for Math

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1a. Show that the set of functions $f: \mathbb{R} \to \mathbb{R}$ which are differentiable between 0 and 1 form an infinite dimensional vector space V over $\mathbb{R}$. (3 pts.)

1b. Find an explicit subspace W of V of codimension 1. (4 pts.)

2. Let V be a vector space over a field K. Let $(v_i)_{i \in I}$ be a basis of V. For $j \in I$, define a function v_j^* from V into K as follows: For all $v \in V$, if $v = \sum_{i\in I} \alpha_i v_i$ then $v_j^*(v) = \alpha_j$. I.e. v_j^* is the j^{th} projection map.

2a. Show that the set of linear maps from V into K form a vector space V^*. (2 pt.)

2b. Show that $v_j^* \in V^*$ for all $j \in J$. (1 pt.)

2c. Show that the linear maps v_j^* are linearly independent. (4 pts.)

2d. Assume V is finite dimensional. Let $f \in V^*$. Show that (4 pts.)

$$f = \sum_{i\in I} f(v_i) v_i^*$$

Conclude that the set $(v_i^*)_{i\in I}$ form a basis of V^* when V is finite dimensional. (1 pt.)

2e. Assume V is infinite dimensional. Define $f: V \to K$ as follows: For $v \in V$, if

$$v = \sum_{i\in I} \alpha_i v_i \text{, then } f(v) = \sum_{i\in I} \alpha_i .$$

Show that f is not in the subspace of V^* generated by $(v_i^*)_{i\in I}$. (7 pts.)

2f. V^{**} denotes $(V^*)^*$. For $v \in V$, define $v^{**} \in V^{**}$ as follows: For $f \in V^*$,

$$v^{**}(f) = f(v).$$

Show that v^{**} is really an element of V^{**}. (2 pts.)

2g. Show that the map that sends $v \in V$ into $v^{**} \in V^{**}$ is a one-to-one linear map from V into V^{**}. (7 pts.)

2ğ. Show that if V is finite dimensional, V and V^{**} are canonically isomorphic (i.e. there is an isomorphism from V into V^{**} whose definition does not depend on the choice of a basis of V). (4 pts.)

2h. Show that if V is infinite dimensional, the map in **2g** is never onto. (**Hint:** Get your inspiration from **2e**.) (10 pts.)

3. Let V be a finite dimensional vector space over a field K. Let $f: V \times V \to K$ be a bilinear map. Let $(v_i)_{i=1, ..., n}$ be a basis of V, $f(v_i, v_j) = \alpha_{ij}$ and $A(f, v_1, ..., v_n) = (\alpha_{ij})_{ij}$, the $n \times n$ matrix.

3a. If $x \in V$ is written as $a_1v_1 + ... + a_nv_n$, we set $X = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$. Show that $f(x, y) = X^tAY$ where X^t denotes the transpose of X and $A = A(f, v_1, ..., v_n)$. (Here we identified the 1×1 matrices with their entries). (5 pts.)

3b. Let $P \in GL_n(K)$. Show that $A(f, Pv_1, ..., Pv_n) = P^tA(f, v_1, ..., v_n)P$. (5 pts.)

3c. A bilinear map f is called **symmetric** if $f(x, y) = f(y, x)$ for all $x, y \in V$. It is called **antisymmetric** if $f(x, y) = -f(y, x)$ for all $x, y \in V$. Let $\mathbf{L}(V)$, $\mathbf{L}_s(V)$ and $\mathbf{L}_a(V)$ be the set of bilinear, symmetric bilinear and antisymmetric bilinear maps on $V \times V$ respectively. Show that $\mathbf{L}(V)$, $\mathbf{L}_s(V)$ and $\mathbf{L}_a(V)$ are all vector spaces and that if char(K) $\neq 2$, then $\mathbf{L}(V) = \mathbf{L}_s(V) \oplus \mathbf{L}_a(V)$. (5 pts.)

3d. Let $f \in \mathbf{L}_s(V)$ and $A \subseteq V$. Define $A^{\perp} = \{v \in V : f(v, a) = 0 \text{ all } a \in A\}$. Show that $A^{\perp}$ is a subspace of V and that $A^{\perp\perp\perp} = A^{\perp}$ for all $A \subseteq V$. (5 pts.)

3e. Let $f \in \mathbf{L}_s(V)$. We say that f is **nondegenerate** if $V^{\perp} = 0$. Show that f is nondegenerate iff A is invertible. (6 pts.)

3f. Let $f \in \mathbf{L}_s(V)$ be nondegenerate. We say that $u \in GL_n(K)$ respects f iff

$$f(x, y) = f(u(x), u(y))$$

for all $x, y \in V$. Show that the set $O(f)$ of linear maps that respect f form a subgroup of $GL_n(K)$. (8 pts.)

3g. Let $n = 2$ and define $f((x_1, y_1)(x_2, y_2)) = x_1x_2 + y_1y_2$. Show that f is a nondegenerate symmetric bilinear map. Find explicitely the elements of $O(f)$. What can you say about the group structure of $O(f)$? (2 + 5 + 5 pts.)

3h. Let V be the set of integrable functions from the interval (0, 1) into $\mathbb{R}$. Show that $f(v, w) = \int_0^1 v(t)w(t)dt$ is a nondegenerate symmetric bilinear map. (You have to show first that vw is integrable). (5 pts.)