Linear Algebra for Math

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1. Let

$$v_1 = (-1,2,1,0)$$
$$v_2 = (0,2,1,1)$$
$$v_3 = (1,0,1,-1)$$
$$v_4 = (0,1,1,1)$$

1a. Show that these vectors form a basis of $\mathbb{R}^4$. (5 pts.)

1b. Write an arbitrary vector (a, b, c, d) as a linear combination of these four vectors. (5 pts.)

2a. Find the inverse of the matrix (10 pts.)

$$\begin{pmatrix} 0 & 1 & 0 & 1 \\ 2 & 0 & 1 & 2 \\ 2 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 \end{pmatrix}$$

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3. Let

$$A = \begin{pmatrix} 4 & -4 & 2 \\ 2 & -2 & 2 \\ -2 & 1 & 1 \end{pmatrix}$$

Find A^{333}. (15 pts.)

4a. Show that the set of functions $f : \mathbb{R} \to \mathbb{R}$ which are integrable between 0 and 1 form an infinite dimensional vector space V over $\mathbb{R}$. (10 pts.)

4b. Find a subspace W of V such that $\dim(V/W) = 1$. (10 pts.)

5. Let V be a vector space over $\mathbb{R}$. Let $(v_i)_{i \in I}$ be a basis of V. For $j \in I$, define a function v_j^* from V into $\mathbb{R}$ as follows: For all $v \in V$,

$$v = \sum_{i \in I} \alpha_i v_i$$

then $v_j^*(v) = \alpha_j$.

5a. Show that the set of linear maps from V into $\mathbb{R}$ form a vector space V^*. (10 pts.)

5b. Why is v_j^* well-defined? (5 pts.)

5c. Show that $v_j^* \in V^*$ for all $j \in J$. (5 pts.)

5d. Show that the linear maps v_j^* are linearly independent. (10 pts.)

5e. Assume V is finite dimensional. Let $f \in V^*$ and set $f(v_i) = \beta_i \in \mathbb{R}$. Show that

$$f = \sum_{i \in I} \beta_i v_i *$$

Conclude that the set $(v_i*)_{i \in I}$ form a basis of $V*$. (10 pts.)

5f. Assume V is infinite dimensional. Define f: $V \to \mathbb{R}$ as follows: For $v \in V$, if

$$v = \sum_{i \in I} \alpha_i v_i \text{ , then } f(v) = \sum_{i \in I} \alpha_i \text{ .}$$

Show that f is not in the subspace of $V*$ generated by $(v_i*)_{i \in I}$. (10 pts.)