Linear Algebra

Resit
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Ali Nesin

1. Let $V = \{f : \mathbb{R} \rightarrow \mathbb{R} : f \text{ is differentiable}\}$. We know (from calculus) that V is a vector space over $\mathbb{R}$. Show that the set $(\sin^n x)_n$ is a linearly independent subset of V. (10 pts.)

2. Show that $\mathbb{R} \approx \mathbb{R} \times \mathbb{R}$ as vector spaces over $\mathbb{Q}$. (5 pts.)

3. Let V be a vector space and W a subspace of V. Show that W has a complement in V (i.e. a subspace W_1 of V such that $V = W + W_1$ and $W \cap W_1 = \{0\}$). (10 pts.)

4. Let V be a vector space over a field K. Let $f \in \text{End}_K(V)$ be such that $f \circ f = f$.

4a. Show that $v - f(v) \in \text{Ker}(f)$ for all $v \in V$. (1 pt.)

4b. Show that $\text{Ker}(f) \cap \text{Im}(f) = \{0\}$. (3 pts.)

4c. Show that $V \approx \text{Ker}(f) \times \text{Im}(f)$ (8 pts.)

4b. Show that f is the identity map on $\text{Im}(f)$ and that it is the zero map on $\text{Ker}(f)$. (2 pts.)

5. Recall that $\text{GL}_n(\mathbf{F}_q)$ is the group of invertible $n \times n$ matrices over the field $\mathbf{F}_q$, $\text{SL}_n(\mathbf{F}_q)$ is the group of $n \times n$ matrices of determinant 1 over the field $\mathbf{F}_q$, $\text{PGL}_n(\mathbf{F}_q) = \text{GL}_n(\mathbf{F}_q)/(\mathbf{F}_q^* \text{Id}_n)$ and $\text{PSL}_n(\mathbf{F}_q) = \text{SL}_n(\mathbf{F}_q)/(\mathbf{F}_q^* \text{Id}_n \cap \text{SL}_n(\mathbf{F}_q))$.

5a. Find the number of elements of $\text{GL}_n(\mathbf{F}_q)$, $\text{SL}_n(\mathbf{F}_q)$, $\text{PGL}_n(\mathbf{F}_q)$ and $\text{PSL}_n(\mathbf{F}_q)$. (10 pts.)

5b. Let p be a prime number. Does $\text{PSL}_n(\mathbf{F}_q)$ has an element of order p? (4 pts.)

5c. Assume $n > 1$ and that p divides q. Find an element of order p of $\text{PSL}_n(\mathbf{F}_q)$. (5 pts.)

6. Let $k \leq K \leq L$ be field extensions. Show that $\dim_k(L)$ is finite if and only if $\dim_k(K)$ and $\dim_K(L)$ are finite. Show that in that case we have $\dim_k(L) = \dim_k(K)\dim_K(L)$. (10 pts.)

7a. Let A be an abelian group of prime exponent p. Show that A is, in a natural way, a vector space over $\mathbf{F}_p$. (10 pts.)

7b. Show that, up to isomorphism, there is a unique abelian group of prime exponent of a given cardinality. (5 pts.)

8a. Let A be a torsion-free divisible abelian group. Show that A is, in a natural way, a vector space over $\mathbb{Q}$. (10 pts.)

8b. Show that, up to isomorphism, there is a unique torsion-free divisible abelian group of a given infinite cardinality. (5 pts.)