Linear Algebra

Resit
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Ali Nesin

1. Find

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 1 & 1 & 0 \end{pmatrix}^{-1}$$

2. Find

$$\begin{pmatrix} 3 & 2 & 0 \\ -1 & 0 & 0 \\ -1 & -1 & 1 \end{pmatrix}^{n}$$

for $n \in \mathbf{Z}$.

3. Let $V = \{f : \mathbb{R} \to \mathbb{R} : f \text{ is differentiable}\}$. We know (from calculus) that V is a vector space over $\mathbb{R}$. Show that the set $(\sin^n x)_n$ is a linearly independent subset of V.

4a. Is $\mathbb{R}$ a vector space over $\mathbb{Q}$?
4b. Is $\mathbb{Q}$ a vector space over $\mathbb{Z}$?

5. Let $V = \{(x, y, z, t, u) \in \mathbb{R}^5 : x + y + z - t + u\}$. Let
$$W = \{(x, y, z, t, u) \in V : 2x - 2y + z - t + u = 0 \text{ and } 2x - 3t = 0\}.$$
We know that V is a vector space over $\mathbb{R}$ and that W is a subspace of V. Find a basis of V, W and V/W.

6. Let V be a vector space over a field K. Let $f \in \mathrm{End}_K(V)$ be such that $f \circ f = f$.
6a. Show that $v - f(v) \in \mathrm{Ker}(f)$ for all $v \in V$.
6b. Show that $\mathrm{Ker}(f) \cap \mathrm{Im}(f) = \{0\}$.
6c. Show that $V \approx \mathrm{Ker}(f) \times \mathrm{Im}(f)$
6b. Show that f is the identity map on $\mathrm{Im}(f)$ and that it is the zero map on $\mathrm{Ker}(f)$.

7. Recall that $\mathrm{GL}_n(\mathbf{F}_q)$ is the group of invertible $n \times n$ matrices over the field $\mathbf{F}_q$, $\mathrm{SL}_n(\mathbf{F}_q)$ is the group of $n \times n$ matrices of determinant 1 over the field $\mathbf{F}_q$, $\mathrm{PGL}_n(\mathbf{F}_q) = \mathrm{GL}_n(\mathbf{F}_q)/(\mathbf{F}_q^{*}\,\mathrm{Id}_n)$ and $\mathrm{PSL}_n(\mathbf{F}_q) = \mathrm{SL}_n(\mathbf{F}_q)/(\mathbf{F}_q^{*}\,\mathrm{Id}_n \cap \mathrm{SL}_n(\mathbf{F}_q))$.
7a. Find the number of elements of $\mathrm{GL}_n(\mathbf{F}_q)$, $\mathrm{SL}_n(\mathbf{F}_q)$, $\mathrm{PGL}_n(\mathbf{F}_q)$ and $\mathrm{PSL}_n(\mathbf{F}_q)$.
7b. Let p be a prime number. Does $\mathrm{PSL}_n(\mathbf{F}_q)$ has an element of order p?
7c. Assume $n > 1$ and that p divides q. Find an element of order p of $\mathrm{PSL}_n(\mathbf{F}_q)$.