Well-Ordered Sets, Ordinals and Cardinals

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Definitions. A set together with a binary relation $<$ is called a **partially ordered set (poset** in short) if

$\forall x \neg(x < x)$

$\forall x \forall y \forall z ((x < y \wedge y < z) \rightarrow x < z)$

If in a poset any two elements can be compared, i.e. if

$\forall x \forall y \, x < y \vee x = y \vee y < x$

then we say that the set is **totally ordered**.

If every nonempty subset of a totally ordered set has a least element, then we say that the set is **well-ordered**.

Note that every nonempty well-ordered set has a least element, but not necessarily a largest element.

Examples.

1. Any natural number n and the set $\mathbb{N}$ (or ω) are well-ordered sets, on the other hand, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ are not well-ordered with respect to their natural order.
2. If X is a well-ordered set and ∞ is a new element not in X, by putting the element ∞ to the very end of X we obtain a new well-ordered set $X \cup \{\infty\}$. If $x \notin x$, one usually takes x as the new element[2]. In that case we set $X^+ = X \cup \{X\}$. Note that X^+ has a largest element, namely X. Since $\omega \notin \omega$, starting from ω, this gives us new well-ordered sets.
3. Any subset of a well-ordered set is also well-ordered by the induced order.
4. Let X and Y be two well-ordered sets. Order the set $(X \times \{0\}) \cup (Y \times \{1\})$ as follows: For all $x, x_1, x_2 \in X$ and for all $y, y_1, y_2 \in Y$

   $(x_1, 0) < (x_2, 0)$ iff $x_1 < x_2$.

   $(y_1, 1) < (y_2, 1)$ iff $y_1 < y_2$.

   $(x, 0) < (y, 1)$.

   This is a well-order. We may denote this well-ordered set by $X + Y$.
5. Let X and Y be two well-ordered sets. Order the set $X \times Y$ as follows:

   $$(a, b) < (a', b') \text{ iff either } b < b' \text{ or } (b = b' \text{ and } a < a').$$

   This is a well-order. We may denote this well-ordered set by $X \times Y$ or simply by XY.

Remarks. Let X be a well-ordered set.

1. For $x \in X$, we set $s(x) = \{y \in X : y < x\}$. A subset A of a well-ordered set X is called an **initial segment** if for $a \in A$, $s(a) \subseteq A$. Note that $s(x)$ is always an initial segment and all initial segments except X are of this form, in other words, x is the least element of $X \setminus s(x)$.
2. If $x \in X$ is not the largest element, then the least element of $\{y \in X : x < y\}$ is the immediate successor of x. Thus in a well-ordered set, every element,

[1] Mainly from Suppes, **Axiomatic Set Theory**.

[2] The **Axiom of Regularity** states that no set is its own element.

except may be the last one, has an immediate successor. But not every element has an immediate predecessor.
3. If every countable subset of a totally ordered set X is well-ordered, then X is well-ordered.
4. $\omega^+ + \omega^+ = (\omega 2)^+$.
5. $2\omega = \omega$. More generally $n\omega = \omega$ for $n > 0$.

Transfinite Induction. *Let X be a well-ordered set and $A \subseteq X$. Assume that for any element x of X, $x \in A$ whenever $s(x) \subseteq A$. Then $A = X$.*

Proof. Assume $A \neq X$. Let x be the least element of $X \setminus A$. Then $s(x) \subseteq A$. Hence $x \in A$, a contradiction. QED

Definition. An **ordinal** is a well-ordered set α such that $\beta = s(\beta)$ for all $\beta \in \alpha$.

Remarks.

1. Every element of an ordinal is the set of its predecessors and is an ordinal itself. Thus every element of an ordinal is also a subset of this ordinal. But not every subset of an ordinal is an ordinal. On the other hand, every initial segment of an ordinal is an ordinal.
2. An ordinal is a set α well-ordered by the relation $\in$, i.e. the binary relation $<$ on α defined by "$\beta < \gamma$ iff $\beta \in \gamma$" well-orders α.
3. The least element of an ordinal is 0, because if o is the least element of an ordinal, $o = s(o) = \varnothing = 0$. The second element is necessarily 1 because if x is the second element, then, $x = s(x) = \{0\} = 1$. The third element must be 2, etc.
4. If α is an ordinal, then $\alpha \notin \alpha$, because otherwise, since $\alpha \in \alpha$, $\alpha \in \alpha = s(\alpha)$ and $\alpha < \alpha$, contradicting the fact that an ordinal is a poset. Thus α^+ is also an ordinal where α is the largest element.
5. If α is an ordinal and $\beta \in \alpha$, then either $\beta^+ \in \alpha$ or $\beta^+ = \alpha$.
6. If α and β are ordinals and $\beta \subset \alpha$, then $\beta \in \alpha$ and β is an initial segment of α.
7. For any two ordinals α and β, either $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$. Thus every set of ordinals is well-ordered by the membership relation and the union of a set of ordinals is an ordinal.
8. If two ordinals α and β are isomorphic as well-ordered sets, then $\alpha = \beta$.
9. Every well-ordered set is isomorphic to a unique ordinal.
10. Let α be an ordinal and $X \subseteq \alpha$. Then the set $\{y \in \alpha\colon y \in x \text{ for some } x \in X\}$ is also an ordinal.

A nonzero ordinal λ is called a **limit ordinal** if it has no predecessor, i.e. if there is no α such that $\alpha^+ = \lambda$.

Remarks.

1. ω is the first limit ordinal.
2. A limit ordinal is its own union. On the other hand the union of α^+ is only α. Thus a nonzero ordinal is a limit ordinal iff it is equal to the union of its elements.

Transfinite Induction. *Let* α *be an ordinal. Let* $A \subseteq \alpha$ *be such that*

i) $0 \in A$,

ii) *if* $\beta \in A$ *then* $\beta^+ \in A$,

iii) *If* $\gamma \in A$ *for all elements* γ *of a limit ordinal* $\beta \in \alpha$ *then* $\beta \in A$.

Then $A = \alpha$.

Definition. We define the **addition** of ordinals as follows:

$$\alpha + 0 = \alpha$$
$$\alpha + \beta^+ = (\alpha + \beta)^+$$
$$\alpha + \lambda = \bigcup_{\beta<\lambda} \alpha+\beta \text{ if } \lambda \text{ is a limit ordinal}$$

Remarks.

1. $\alpha + \beta$ is an ordinal.
2. $\alpha + 1 = \alpha + 0^+ = (\alpha + 0)^+ = \alpha^+$.
3. The addition coincides with the usual addition on ω.
4. $n + \omega = \omega$ for $n \in \omega$.
5. $0 + \alpha = \alpha$.
6. If $\beta < \gamma$ then $\alpha + \beta < \alpha + \gamma$. (On the other hand, $1 + \omega = 2 + \omega$). Thus if $\alpha + \beta = \alpha + \gamma$ then $\beta = \gamma$.
7. If $\alpha \le \beta$ then $\alpha + \gamma \le \beta + \gamma$. Thus $\beta \le \alpha + \beta$.
8. If λ is a limit ordinal, then so is $\alpha + \lambda$. Does the converse hold?
9. $\omega + \omega$ is the second limit ordinal.
10. If $\alpha \le \beta$ then there is a unique γ such that $\alpha + \gamma = \beta$.
11. $\alpha + \beta$ is the first ordinal greater than $\alpha + \delta$ for all $\delta < \beta$.
12. $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$.
13. $(\omega + n) + (\omega + m) = (\omega + \omega) + m$

Definition. We define the multiplication of ordinals as follows:

$$\alpha \times 0 = 0$$
$$\alpha \times \beta^+ = (\alpha \times \beta) + \alpha$$
$$\alpha \times \lambda = \bigcup_{\beta<\lambda} \alpha\times\beta \text{ if } \lambda \text{ is a limit ordinal}$$

Remarks.

1. We may write $\alpha\beta$ for $\alpha \times \beta$.
2. $\alpha\beta$ is an ordinal.
3. The multiplication of natural numbers corresponds to the usual multiplication.
4. $0\alpha = 0$.
5. $\alpha 1 = 1\alpha = \alpha$.
6. $\alpha 2 = \alpha + \alpha$, but $2\omega = \omega \ne \omega + \omega$. In general $n\omega = \omega$.
7. $n(\omega + m) = \omega + nm$.
8. $(\omega + 1)2 = \omega 2 + 1$
9. If $\alpha > 0$ and $\beta < \gamma$ then $\alpha\beta < \alpha\gamma$. In particular if $\alpha \ne 0$ and $\beta \ne 0$ then $\alpha\beta \ne 0$ and if $\alpha\beta = \alpha\gamma$ and $\alpha \ne 0$ then $\beta = \gamma$.
10. If $\alpha > 0$ and β is a limit ordinal, then $\alpha\beta$ and $\beta\alpha$ are limit ordinals.
11. $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$. On the other hand, $1\omega + 1\omega = \omega + \omega \ne \omega = 2\omega = (1 + 1)\,\omega$.

12. $\alpha(\beta\gamma) = (\alpha\beta)\gamma$
13. If $\alpha\beta < \alpha\gamma$ and $\alpha \neq 0$ then $\beta < \gamma$.
14. If $\beta\alpha < \gamma\alpha$ and $\alpha \neq 0$ then $\beta < \gamma$.
15. If γ is a limit ordinal then $n\gamma = \gamma$ for $n > 0$.
16. If $\gamma < \alpha\beta$ then there is a unique $\alpha_1 < \alpha$ and a unique $\beta_1 < \beta$ such that $\gamma = \alpha\beta_1 + \alpha_1$.
17. If $\alpha \neq 0$ then for any β there is a unique γ and a unique $\delta < \alpha$ such that $\beta = \alpha\gamma + \delta$.
18. If α is a limit ordinal then there is a unique β such that $\alpha = \omega\beta$.

Definition: We define the exponentiation of ordinals as follows:

$$\alpha^0 = 1$$
$$\alpha^{\beta+1} = \alpha^\beta\,\alpha$$
$$\alpha^\lambda = \bigcup_{\beta<\lambda} \alpha^\beta \text{ if } \lambda \text{ is a limit ordinal and } \alpha \neq 0$$
$$0^\lambda = 0 \text{ if } \lambda \text{ is a limit ordinal.}$$

Remarks.

1. α^β is an ordinal.
2. Exponention coincides with the usual one on ω.
3. $1^\alpha = 1$.
4. $2^\omega = \omega$. More generally $n^\omega = \omega$ if $n \geq 2$.
5. $\omega < \omega^2 < \omega^3 < ... < \omega^\omega$.
6. $\alpha^2 = \alpha\alpha$, $\alpha^3 = \alpha\alpha\alpha$, etc.
7. $\alpha^{\beta+\gamma} = \alpha^\beta\alpha^\gamma$.
8. $(\alpha^\beta)^\gamma = \alpha^{\beta\gamma}$. On the other hand $(\omega 2)^2 \neq \omega^2 2^2$.
9. $2^{(\omega^{n+1})} = \omega^{(\omega^n)}$.
10. If $\alpha > 1$ and $\beta < \gamma$ then $\alpha^\beta < \alpha^\gamma$.
11. If $\alpha > 1$ and $\beta > 1$ then $\alpha + \beta \leq \alpha\beta \leq \alpha^\beta$.
12. (Sierpinski 1950) For any ordinal number $\mu \geq 1$ there are α, β, γ such that $\alpha^\mu + \beta^\mu = \gamma^\mu$. Indeed, if μ is not a limit ordinal, then for $\xi \geq 1$,
$$(\omega^\xi)^\mu + (\omega^\xi 2)^\mu = (\omega^\xi 3)^\mu.$$
And if μ is a limit ordinal, then for $\xi \geq 1$,
$$(\omega^\xi)^\mu + (\omega^{\xi\mu})^\mu = (\omega^{\xi\mu} + 1)^\mu.$$

Definition. An ordinal which is not in bijection with one of its elements is called a **cardinal (number)**.

Remarks.

1. Each natural number and ω are cardinals. $\omega + n$ is not an ordinal for $n > 0$. $\omega + \omega$ and ω^ω are not cardinals neither.
2. Cardinals which are not natural numbers are necessarily limit ordinals. But not every limit ordinal is a cardinal.

If α is an infinite cardinal, then $\alpha\alpha = \alpha$.

7. The purpose of this exercise is to show that there is an uncountable family of subsets of **N** such that any two subsets of the family intersect in a finite set.

We know that **Q** is countable. Let $(a_n)_{n \in \mathbf{N}}$ be an enumeration of **Q** by natural numbers.

7a. Show that for every $r \in \mathbb{R}$, there is a strictly increasing subsequence of $(a_n)_{n \in \mathbf{N}}$ that converges to r.

7b. Show that there is a set **S** of strictly increasing Cauchy subsequences of $(a_n)_{n \in \mathbf{N}}$ such that for every $r \in \mathbb{R}$, there is a unique element of **S** that converges to r.

Call this subsequence $s_r = (a_{n(r)})_{n(r)}$. Let $I(r)$ be the set of $n(r)$. Note that $I(r)$ is an infinite subset of $\mathbb{N}$.

7c. Show that for two distinct real numbers r, r', $I(r) \cap I(r')$ is finite.

7d. Deduce that there is an uncountable family of subsets of $\mathbb{N}$ whose two by two intersections are finite.