Quiz

Set Theory (Math 111)
May 5th,1999
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General Hint: Draw pictures.

Recall that a set X together with a binary relation $<$ is called **well-ordered** if the following conditions are satisfied:

i) For all $x, y, z \in X$, if $x < y$ and $y < z$ then $x < z$.
ii) For all $x \in X$, $\neg(x < x)$.
iii) For all $x, y \in X$, either $x < y$ or $x = y$ or $y < x$.
iv) Every nonempty subset of X has a least element.

A set is called **well-orderable** if there is a binary relation $<$ that well-orders X.

I. Let X be a set well-ordered by the binary relation $<$. Let $x \in X$ and define $[0, x)$ to be the subset $\{y \in X: y < x\}$ of X. Consider the following property of a subset Y of X:

$$\text{For all } a \in Y, \text{ if } b < a \text{ then } b \in Y. \qquad (*)$$

Show that a subset Y of X satisfies (*) and $Y \neq X$ if and only if $Y = [0, x)$ for some $x \in X$.

II. Let X be a well-orderable set. Let Y be any subset of X. Show that Y is also well-orderable.

III. Let X be a well-orderable set. Let Y be any set. Show that if there is a one-to-one mapping from Y into X then Y is also well-orderable.

IV. Is it true that every nonempty subset of a well-ordered set has a greatest element?

V. Let X be a set well-ordered by the relation $<$. Let $y \notin X$ and $Y = X \cup \{y\}$. Define a binary relation $<'$ on Y by

$$a <' b \text{ iff } a \in X \wedge ((b \in X \wedge a < b) \vee b = y).$$

Va. Show that $<'$ well-orders Y.

Vb. Show that Y has a greatest element.

VI. Let the binary relation $<$ well-order the set X. Let $\pi: X \to X$ be an **order-preserving** function, i.e. a function such that for all $x, y \in X$, $x < y$ iff $\pi(x) < \pi(y)$. Show that for all $x \in X$, $x \leq \pi(x)$ (i.e. either $x < \pi(x)$ or $x = \pi(x)$).

VII. Is it true that an order preserving map is always one-to-one?

VIII. Let the binary relation $<$ well-order the set X and let $x \in X$. Let $Z = [0, x)$. Show that there is no order-preserving onto map from X into Z.

IX. Let the binary relation $<$ well-order the set X. Let $\varphi(x)$ be a formula. Assume that,

$$\text{For every } x \in X, \text{ if } \varphi(y) \text{ for every } y < x, \text{ then } \varphi(x).$$

Show that,

$$\text{For every } x \in X, \varphi(x).$$