Second Midterm
Math 111

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Notation: **N** denotes the set of natural numbers. $\mathbb{Z}$ denotes the set of integers. $\mathbb{R}$ denotes the set of real numbers. If X is a set, $\wp(X)$ denotes the set of subsets of X and Id_X denotes the identity map on X, i.e. $\mathrm{Id}_X(x) = x$ for all $x \in X$. The symbol $f \circ g$ denotes the composition of the functions f and g whenever it makes sense; thus $(f \circ g)(x) = f(g(x))$. The symbol f^2 denotes the function $f \circ f$ whenever it makes sense.

1. Write all the elements of $\wp(\wp(\{1,2\}))$.

2. Does the following sets have a maximal and a minimal element (for the inclusion)?

a) $\{x \in \wp(\mathbb{N}): 1 \in x, 2 \notin x\}$

b) $\{x \in \wp(\mathbb{N}) : \exists n \in \mathbb{N}\ \forall m > n\ (m \notin x)\}$

c) $\{x \in \wp(\wp(\mathbb{N})): a\ \mathbb{N} \in \boldsymbol{x} \text{ for all } a \in \mathbf{N}\}$

3. Find all the bijections f of the set 4 for which $f^3 = \mathrm{Id}_4$.

4. Find all the bijections $f : \mathbb{N} \to \mathbb{N}$ for which

$$x < y \Rightarrow f(x) < f(y)$$

for all $x, y \in \mathbb{N}$.

5. Let $f : \mathbb{N} \to \mathbb{Z}$ be given by $f(x) = x - x^2$ and $g: \mathbb{Z} \to \mathbb{R}$ be given by $g(x) = x/(1+x^2)$. Compute $(g \circ f)(3)$. For what values of x, does $(g \circ f)(x) \geq 0$?

6. Let X be a set and $f: X \to X$ be a function such that f^2 is a bijection. Show that f is a bijection.

7. Give an example of a nonconstant function $f : \mathbb{N} \to \mathbb{N}$ such that $f \neq \mathrm{Id}_{\mathbb{N}}$ and $f^2 = f$.

8. Show that for any nonzero natural number n,

$$\frac{1}{1}\frac{1}{3} + \frac{1}{3}\frac{1}{5} + \frac{1}{5}\frac{1}{7} + \ldots + \frac{1}{2n-1}\frac{1}{2n+1} = \frac{n}{2n+1}$$

9. Let A, B, C be three sets. Show that $(A \cap C) \setminus (B \cap C) \subseteq A \setminus B$.

10. Let X be a set. A set Y is said to be a **choice set** for X, if for any $x \in X$ there is a unique element $y \in Y$ such that $y \in x$. Find a choice set for $X = \{\{0,1\},\{1,2\}\}$. Find a set X which has no choice function.