Set Theory (Orders)

December 2005

Ali Nesin

Recall that a **preorder** on a nonempty set X is a subset A of $X \times X$ such that

a) For any $x \in X$, $(x, x) \notin A$.

b) For any $x, y, z \in X$, if $(x, y) \in A$ and $(y, z) \in A$, then $(x, z) \in A$.

If A is a preorder on a set X, instead of "$(x, y) \in A$" one often writes xAy, and in this case we say that (X, A) is a preorder. Instead of A, it is customary to use a more common symbol such as $<$ or $\prec$.

Let $(X, <)$ and $(Y, \prec)$ be two preorders. A **morphism** from $(X, <)$ into $(Y, \prec)$ is a map f from X into Y such that for any $x_1, x_2 \in X$, $x_1 < x_2$ iff $f(x_1) \prec f(x_2)$.

1. Show that a morphism is necessarily one-to-one.

2. Show that the composition of morphisms is a morphism.

A morphism f from $(X, <)$ into $(Y, \prec)$ is called an **isomorphism** if the map is onto. The identity map is clearly an isomorphism.

3. Show that if f is an isomorphism from $(X, <)$ into $(Y, \prec)$ then f^{-1} is an isomorphism from $(Y, \prec)$ into $(X, <)$ is an isomorphism.

If there is an isomorphism between the preorders $(X, <)$ and $(Y, \prec)$ they are said to be **isomorphic**.

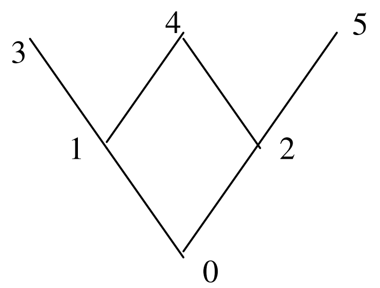
4. How many nonisomorphic preorders are there on a set of three elements?

5. How many nonisomorphic preorders are there on a set of four elements?

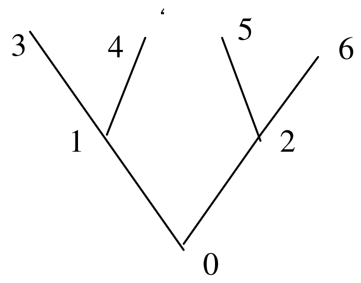
6. Let $\sigma \in \mathrm{Sym}(B)$. Show that σ gives rise (naturally) to an automorphism of the preorder $(\wp(B), \subset)$. Conversely, show that every automorphism of the preorder $(\wp(B), \subset)$ is of this form.

An **automorphism** of a preorder is an isomorphism from the preorder into itself.

7. Find all the automorphisms of the following preorder



8. Find all the automorphisms of the following preorder



A **total order** is an order where any two elements are comparable, i.e. for any x, $y \in X$, either $x < y$ or $x = y$ or $y < x$ (only one of the three relations may hold). For example, the natural orders on $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ are total orders.

9. Show that any two total orders on a finite set X are isomorphic.

10. Find all the automorphisms of $(\mathbb{N}, <)$. Here $<$ is the usual ordering on $\mathbb{N}$.

11. Find all the automorphisms of $(\mathbb{Z}, <)$. Here $<$ is the usual ordering on $\mathbb{Z}$.

12. Show that the usual total orders $(\mathbb{N}, <)$ and $(\mathbb{N} \setminus \{0\}, <)$ are isomorphic.

13. Show that the usual total orders $(\mathbb{Z}, <)$ and $(\mathbb{Z} \setminus \{5\}, <)$ are isomorphic.

14. Show that the usual total orders $(\mathbb{N}, <)$ and $(\mathbb{Z}, <)$ are not isomorphic.

15. Define a total order on the set $\mathbb{N} \cup \{\infty\}$ by extending the usual total order of $\mathbb{N}$ by adding $n < \infty$ for all $n \in \mathbb{N}$. Show that $(\mathbb{N} \cup \{\infty\}, <)$ and $(\mathbb{N}, <)$ are not isomorphic.