0a. Show that for any natural number \( m \) if \( n \in m \) then \( n \subseteq m \). (4 pts.)

**Proof:** By induction on \( m \).

If \( m = 0 = \emptyset \), there is no such \( n \), so this case is trivial.

Assume the statement for \( m \) and let \( n \in m + 1 = m \cup \{m\} \). Then either \( n \in m \) or \( n = m \).

In the first case \( n \subseteq m \) by induction. Thus in both cases \( n \subseteq m \).

Hence \( n \subseteq m \subseteq m \cup \{m\} = m + 1 \).

0b. Show that for any natural number \( n \), \( n \notin n \). (4 pts.)

**Proof:** Clearly \( 0 \notin \emptyset = 0 \). Assume \( n \notin n \). If \( n + 1 \in n + 1 = n \cup \{n\} \), then \( n + 1 \in n \) or \( n + 1 = n \). In both cases \( n + 1 \subseteq n \). Thus \( n \cup \{n\} \subseteq n \). Hence \( n \in n \), a contradiction.

0c. Let \( \omega \) be the set of natural numbers. Show that \( \omega \neq n \) for any \( n \in \omega \). (3 pts.)

**Proof:** By induction on \( n \). If \( n = 0 = \emptyset \), since \( 0 = \emptyset \in \omega \), clearly \( \omega \neq 0 \).

Assume \( n \neq \omega \) but that \( n + 1 = \omega \). Then \( n + 1 \subseteq \omega \) by induction. Thus \( n + 1 \subseteq n + 1 = n \cup \{n\} \subseteq n \). Hence \( n \in n \), a contradiction.

0d. Show that \( \omega \notin \omega \). (2 pts.)

**Proof:** Assume \( \omega \in \omega \). Then \( \omega = n \) for some \( n \in \omega \). Contradicting 0c.

0e. Let \( \omega + 1 = \omega \cup \{\omega\} \). Show that \( \omega + 1 \notin \omega + 1 \). (3 pts.)

**Proof:** Assume \( \omega + 1 \in \omega + 1 = \omega \cup \{\omega\} \). Then either \( \omega + 1 = \omega \) or \( \omega + 1 = \omega \), i.e. either \( \omega \cup \{\omega\} \in \omega \) or \( \omega \cup \{\omega\} = \omega \). In both cases \( \omega \in \omega \), contradicting 0d.

1. A set \( X \) is called \( \in \)-complete if every element of \( X \) is a subset of \( X \).

1a. Show that a set \( X \) is \( \in \)-complete if and only if \( \cap X \subseteq X \). (2 pts.)

**Proof:** (\( \Rightarrow \)) Let \( a \in \cap X \). Then there is a \( b \in X \) such that \( a \in b \). Since \( X \) is \( \in \)-complete, \( b \subseteq X \). Thus \( a \in b \subseteq X \) and so \( a \in X \).

(\( \Leftarrow \)) Let \( a \in X \). Then \( a \subseteq \cup X \subseteq X \).

1b. Show that if \( X \) is complete, then \( X \cup \{X\} \) is \( \in \)-complete. (3 pts.)

**Proof:** \( \cup (X \cup \{X\}) = \cup_{x \in X \cup \{X\}} x = (\cup_{x \in X} x) \cup (\cup_{x \in \{X\}} x) = (\cup X) \cup (\cup \{X\}) = X \).

1c. Give infinitely many examples of \( \in \)-complete sets. (2 pts.)

**Answer:** Every natural number is an \( \in \)-complete set. We can prove this by induction on \( n \in \mathbb{N} \) by using 1b.

1d. Show that if \( A \) is a set of \( \in \)-complete sets, then \( \cap A \) and \( \cup A \) are also complete. (2 pts.)

**Proof:** Let \( x \in \cap A \). Then \( x \subseteq y \) for any \( y \in A \). But \( y \) is \( \in \)-complete, so \( x \subseteq y \) for any \( y \in A \). Thus \( x \subseteq \cap A \).
Let \( x \in \bigcup A \). Then \( x \in y \) for some \( y \in A \). But \( y \) is \( \in \)-complete, so \( x \subseteq y \). Thus \( x \subseteq \bigcup A \).

1e. Assume \( \{x\} \) is \( \in \)-complete. What can you say about \( x \)? (3 pts.)
Answer: Since \( x \) is the only element of \( \{x\} \), \( \{x\} \) is \( \in \)-complete if and only if \( x \subseteq \{x\} \) if and only if \( x = \emptyset \) or \( x = \{x\} \) if and only if \( x = \emptyset \) or \( x \in x \).

1f. Let \( X \) be any set. Define \( X_0 = X \) and \( X_{n+1} = X_n \cup (\bigcup X_n) \) for any \( n \in \mathbb{N} \). Let \( X_\omega = \bigcup_{n \in \mathbb{N}} X_n \). Assuming \( X_\omega \) is a set, show that it is the smallest \( \in \)-complete set containing \( X \). (8 pts.)
Proof: Remark first that \( X = X_0 \subseteq X_1 \subseteq X_2 \subseteq \ldots \subseteq X_n \subseteq X_{n+1} \subseteq \ldots \subseteq X_\omega \).
Next we show that \( X_\omega \) is \( \in \)-complete. Let \( x \in X_\omega \). Then \( x \in X_n \) for some natural number \( n \). Hence \( x \subseteq \bigcup X_n \subseteq X_{n+1} \subseteq X_\omega \).
Now we show that \( X_\omega \) is the smallest \( \in \)-complete set containing \( X \). Let \( Y \) be any \( \in \)-complete set containing \( X \). Clearly \( X_\omega \subseteq Y \) is equivalent to \( X_n \subseteq Y \) for any natural number \( n \). We will show by induction on \( n \) that \( X_n \subseteq Y \) for any natural number \( n \). If \( n = 0 \) then \( X_n = X_0 = X \subseteq Y \). Assume \( X_n \subseteq Y \). Then \( X_{n+1} = X_n \cup (\bigcup X_n) \subseteq X_n \cup (\bigcup Y) \subseteq X_n \cup Y = Y \).

2. A set \( X \) is \( \in \)-connected if for any two distinct elements \( x, y \) of \( X \), either \( x \in y \) or \( y \in x \).
2a. Give infinitely many examples of \( \in \)-connected sets. (2 pts.)
Answer: Every natural number is \( \in \)-connected.

2b. Show that a subset of an \( \in \)-connected set is \( \in \)-connected. (3 pts.)
Proof: Let \( X \) be \( \in \)-connected. Let \( Y \subseteq X \). Let \( x, y \in Y \) be two distinct elements. Then \( x, y \in Y \). Hence either \( x \in y \) or \( y \in x \).

2c. Show that if \( X \) is \( \in \)-connected, then \( X \cup \{X\} \) is also \( \in \)-connected. (3 pts.)
Proof: Let \( x, y \in X \cup \{X\} \) be two distinct elements. If both \( x \) and \( y \) are elements of \( X \) then either \( x \in y \) or \( y \in x \). If one of them, say \( x = X \), then \( y \in x \).

2d. Assume \( \{x\} \) is \( \in \)-connected. What can you say about \( x \)? (2 pts.)
Proof: Nothing! A singleton set \( \{x\} \) is \( \in \)-connected because it has no two distinct elements.

3. Axiom of Regularity says that every nonempty set \( A \) has an element \( x \) such that \( A \cap x = \emptyset \).
3a. Assuming the Axiom of Regularity show that no set \( x \) is a member of itself. (2 pts.)
Proof: Let \( x \) be any set and let \( A = \{x\} \). By the Axiom of Regularity \( A \cap x = \emptyset \), i.e. \( \{x\} \cap x = \emptyset \), hence \( x \notin x \) (otherwise \( x \in \{x\} \cap x \)).

3b. Assuming the Axiom of Regularity show that there are no sets \( x \) and \( y \) such that \( x \in y \) and \( y \in x \). (2 pts.)
**Proof:** Let $x$ and $y$ be any two sets and let $A = \{x, y\}$. By the Axiom of Regularity either $A \cap x = \emptyset$ or $A \cap y = \emptyset$, say $A \cap x = \emptyset$. Thus $\{x, y\} \cap x = \emptyset$. Since $x \notin x$ by 3a, this means that $y \notin x$.

3c. Assuming the Axiom of Regularity show that there are no sets $x$, $y$ and $z$ such that $x \in y$, $y \in z$ and $z \in x$. (2 pts.)

**Proof:** The same as above with $A = \{x, y, z\}$.

4. An ordinal is an $\in$-complete and $\in$-connected set.

4a. Assuming the Axiom of Regularity show that an ordinal is well-ordered by the relation $\in$. (10 pts.)

**Proof:** Let $X$ be an ordinal.

Let $x \in X$. By 3a, $x \notin x$.

Let $x$, $y$, $z \in X$. Assume $x \in y \in z$. By 3b, $x \notin z$. Since $X$ is $\in$-connected, this implies that either $x \in z$ or $z \in x$. The last one is forbidden by 3c. Hence $x \in z$.

Thus $X$ is partially ordered by the relation $\in$.

It remains to show that any nonempty subset $Y$ of $X$ has a least element. (This implies that $X$ is totally ordered). By the Axiom of Regularity, there is an $x \in Y$ such that $x \cap Y = \emptyset$. Now we show that for any $y \in Y$, either $y = x$ or $x \in y$. Assume $y \neq x$. Then, $x$, $y$ are two distinct elements of $X$ and $X$ is $\in$-connected, either $y \in x$ or $x \in y$. If $y \in x$, then $y \in x \cap Y = \emptyset$, a contradiction. Thus $x \in y$.

4b. Show that if $\alpha$ is an ordinal, so is $\alpha + 1$. (Recall that $\alpha + 1$ is defined to be $\alpha \cup \{\alpha\}$). (3 pts.)

**Proof:** By 1b and 2c.

4c. Show that if $\beta$ is an ordinal, then $\cup (\beta + 1) = \beta$. (5 pts.)

**Proof:** $\cup (\beta + 1) = \cup_{\gamma \in \beta + 1} \gamma = \cup_{\gamma \in \beta} \gamma = (\cup_{\gamma \in \beta} \gamma) \cup \beta = \beta$ because, since $\beta$ is an ordinal, for any $\gamma \in \beta$, $\gamma \subseteq \beta$.

4d. Show that every natural number is an ordinal. (2 pts.)

**Proof:** $0 = \emptyset$ and $\emptyset$ is an ordinal. If $n$ is an ordinal $n + 1$ is an ordinal buy 4c.

4e. Assuming the Axiom of Regularity, show that every ordinal is either $\emptyset$ or contains $\emptyset$ as an element. (3 pts.)

**Proof:** Assume $\alpha \neq \emptyset$ is an ordinal. Then by 4a, $\alpha$ has a least element, say $\beta$. By 4a again, $\beta \cap \alpha = \emptyset$. But, since $\beta \in \alpha$, $\beta \subseteq \alpha$. Hence $\beta = \beta \cap \alpha = \emptyset$.

4f. Show that the set $\omega$ of natural numbers is an ordinal. (2 pts.)

**Proof:** This is clear.

4g. Assuming the Axiom of Regularity, show that every element of an ordinal is an ordinal. (3 pts.)

**Proof:** Let $X$ be an ordinal. Let $x \in X$.

We first show that $x$ is an $\in$-complete set. Let $y \in x$. Let $z \in y$. Then $z \in y \in x$. Hence by 4a, $z \in x$. Thus $y \subseteq x$.

Now we show $x$ is $\in$-connected. Let $y, z \in x$ be two distinct elements of $x$. Since $x \subseteq X$, $y$ and $z$ are elements of $X$. Hence either $y \in z$ or $z \in y$. 


4h. An ordinal $\alpha \neq \emptyset$ is called a **limit ordinal** if $\alpha$ is not of the form $\beta + 1$ for some $\beta \in \alpha$. Show that $\omega$ is a limit ordinal but that no natural number is a limit ordinal. (5 pts.)

**Proof:** Suppose $\omega = \beta + 1$ for some $\beta \in \omega$. Then $\omega = \beta + 1 \in \omega$ and we know this is

4i. Show that if $X$ is a set of ordinals such that for all $\alpha, \beta \in X$ either $\alpha \subseteq \beta$ or $\alpha = \beta$ or $\beta \subseteq \alpha$, then $\bigcup X$ is an ordinal. (5 pts.)