Math 111

Fall 2005 Final

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0a. *Show that for any natural number m if $n \in m$ then $n \subseteq m$.* (4 pts.)

Proof: By induction on m.

If $m = 0 = \emptyset$, there is no such n, so this case is trivial.

Assume the statement for m and let $n \in m + 1 = m \cup \{m\}$. Then either $n \in m$ or $n = m$. In the first case $n \subseteq m$ by induction. Thus in both cases $n \subseteq m$. Hence

$$n \subseteq m \subseteq m \cup \{m\} = m + 1.$$

0b. *Show that for any natural number n, $n \notin n$.* (4 pts.)

Proof: Clearly $0 \notin \emptyset = 0$. Assume $n \notin n$. If $n + 1 \in n + 1 = n \cup \{n\}$, then $n + 1 \in n$ or $n + 1 = n$. In both cases $n + 1 \subseteq n$. Thus $n \cup \{n\} \subseteq n$. Hence $n \in n$, a contradiction.

0c. *Let ω be the set of natural numbers. Show that $\omega \neq n$ for any $n \in \omega$.* (3 pts.)

Proof: By induction on n. If $n = 0 = \emptyset$, since $0 = \emptyset \in \omega$, clearly $\omega \neq 0$.

Assume $n \neq \omega$ but that $n + 1 = \omega$. Then $n + 1 \in \omega = n + 1$, contradicting 0b.

0d. *Show that $\omega \notin \omega$.* (2 pts.)

Proof: Assume $\omega \in \omega$. Then $\omega = n$ for some $n \in \omega$. Contradicting 0c.

0e. *Let $\omega + 1 = \omega \cup \{\omega\}$. Show that $\omega + 1 \notin \omega + 1$* (3 pts.)

Proof: Assume $\omega + 1 \in \omega + 1 = \omega \cup \{\omega\}$. Then either $\omega + 1 \in \omega$ or $\omega + 1 = \omega$, i.e. either $\omega \cup \{\omega\} \in \omega$ or $\omega \cup \{\omega\} = \omega$. In both cases $\omega \in \omega$, contradicting 0d.

1. *A set X is called* $\in$-**complete** *if every element of X is a subset of X.*

1a. *Show that a set X is $\in$-complete if and only if $\cup X \subseteq X$.* (2 pts.)

Proof: ($\Rightarrow$) Let $a \in \cup X$. Then there is a $b \in X$ such that $a \in b$. Since X is $\in$-complete, $b \subseteq X$. Thus $a \in b \subseteq X$ and so $a \in X$.

($\Leftarrow$) Let $a \in X$. Then $a \subseteq \cup X \subseteq X$.

1b. *Show that if X is complete, then $X \cup \{X\}$ is $\in$-complete.* (3 pts.)

Proof: $\cup(X \cup \{X\}) = \cup_{x \in X \cup \{X\}} x = (\cup_{x \in X} x) \cup X = (\cup X) \cup X = X$.

1c. *Give infinitely many examples of $\in$-complete sets.* (2 pts.)

Answer: Every natural number is an $\in$-complete set. We can prove this by induction on $n \in \mathbb{N}$ by using 1b.

1d. *Show that if A is a set of $\in$-complete sets, then $\cap A$ and $\cup A$ are also complete.* (2 pts.)

Proof: Let $x \in \cap A$. Then $x \in y$ for any $y \in A$. But y is $\in$-complete, so $x \subseteq y$ for any $y \in A$. Thus $x \subseteq \cap A$.

Let $x \in \cup A$. Then $x \in y$ for some $y \in A$. But y is $\in$-complete, so $x \subseteq y$. Thus $x \subseteq \cup A$.

1e. *Assume* $\{x\}$ *is* $\in$*-complete. What can you say about* x? (3 pts.)

Answer: Since x is the only element of $\{x\}$, $\{x\}$ is $\in$-complete if and only if $x \subseteq \{x\}$ if and only if $x = \emptyset$ or $x = \{x\}$ if and only if $x = \emptyset$ or $x \in x$.

1f. *Let* X *be any set. Define* $X_o = X$ *and* $X_{n+1} = X_n \cup (\cup X_n)$ *for any* $n \in \mathbb{N}$. *Let* $X_\omega = \cup_{n\in\mathbb{N}} X_n$. *Assuming* X_ω is a set, show that it is the smallest $\in$-complete set containing X. (8 pts.)

Proof: Remark first that $X = X_0 \subseteq X_1 \subseteq X_2 \subseteq ... \subseteq X_n \subseteq X_{n+1} \subseteq ... \subseteq X_\omega$.

Next we show that X_ω is $\in$-complete. Let $x \in X_\omega$. Then $x \in X_n$ for some natural number n. Hence $x \subseteq \cup X_n \subseteq X_{n+1} \subseteq X_\omega$.

Now we show that X_ω is the smallest $\in$-complete set containing X. Let Y be any $\in$-complete set containing X. Clearly $X_\omega \subseteq Y$ is equivalent to $X_n \subseteq Y$ for any natural number n. We will show by induction on n that $X_n \subseteq Y$ for any natural number n. If $n = 0$ then $X_n = X_0 = X \subseteq Y$. Assume $X_n \subseteq Y$. Then

$$X_{n+1} = X_n \cup (\cup X_n) \subseteq X_n \cup (\cup Y) \subseteq X_n \cup Y = Y.$$

2. *A set* X *is* $\in$***-connected*** *if for any two distinct elements* x, y *of* X, *either* $x \in y$ *or* $y \in x$.

2a. *Give infinitely many examples of* $\in$*-connected sets.* (2 pts.)

Answer: Every natural number is $\in$-connected.

2b. *Show that a subset of an* $\in$*-connected set is* $\in$*-connected.* (3pts.)

Proof: Let X be $\in$-connected. Let $Y \subseteq X$. Let $x, y \in Y$ be two distinct elements. Then $x, y \in Y$. Hence either $x \in y$ or $y \in x$.

2c. *Show that if* X *is* $\in$*-connected, then* $X \cup \{X\}$ *is also* $\in$*-connected.* (3 pts.)

Proof: Let $x, y \in X \cup \{X\}$ be two distinct elements. If both x and y are elements of X then either $x \in y$ or $y \in x$. If one of them, say $x = X$, then $y \in x$.

2d. *Assume* $\{x\}$ *is* $\in$*-connected. What can you say about* x? (2 pts.)

Proof: Nothing! A singleton set $\{x\}$ is $\in$-connected because it has no two distinct elements.

3. Axiom of Regularity *says that every nonempty set* A *has an element* x *such that* $A \cap x = \emptyset$.

3a. *Assuming the Axiom of Regularity show that no set* x *is a member of itself.* (2 pts.)

Proof: Let x be any set and let $A = \{x\}$. By the Axiom of Regularity $A \cap x = \emptyset$, i.e. $\{x\} \cap x = \emptyset$, hence $x \notin x$ (otherwise $x \in \{x\} \cap x$).

3b. *Assuming the Axiom of Regularity show that there are no sets* x *and* y *such that* $x \in y$ *and* $y \in x$. (2 pts.)

Proof: Let x and y be any two sets and let $A = \{x, y\}$. By the Axiom of Regularity either $A \cap x = \emptyset$ or $A \cap y = \emptyset$, say $A \cap x = \emptyset$. Thus $\{x, y\} \cap x = \emptyset$. Since $x \notin x$ by 3a, this means that $y \notin x$.

3c. *Assuming the Axiom of Regularity show that there are no sets x, y and z such that $x \in y$, $y \in z$ and $z \in x$.* (2 pts.)
Proof: The same as above with $A = \{x, y, z\}$.

4. *An **ordinal** is an $\in$-complete and $\in$-connected set.*
4a. *Assuming the Axiom of Regularity show that an ordinal is well-ordered by the relation $\in$.* (10 pts.)
Proof: Let X be an ordinal.
Let $x \in X$. By 3a, $x \notin x$.
Let $x, y, z \in X$. Assume $x \in y \in z$. By 3b, $x \neq z$. Since X is $\in$-connected, this implies that either $x \in z$ or $z \in x$. The last one is forbidden by 3c. Hence $x \in z$.
Thus X is partially ordered by the relation $\in$.
It remains to show that any nonempty subset Y of X has a least element. (This implies that X is totally ordered). By the Axiom of Regularity, there is an $x \in Y$ such that $x \cap Y = \emptyset$. Now we show that for any $y \in Y$, either $y = x$ or $x \in y$. Assume $y \neq x$. Then, x, y are two distinct elements of X and X is $\in$-connected, either $y \in x$ or $x \in y$. If $y \in x$, then $y \in x \cap Y = \emptyset$, a contradiction. Thus $x \in y$.

4b. *Show that if α is an ordinal, so is $\alpha + 1$.* (Recall that $\alpha + 1$ is defind to be $\alpha \cup \{\alpha\}$). (3 pts.)
Proof: By 1b and 2c.

4c. *Show that if β is an ordinal, then $\cup(\beta+1) = \beta$.* (5 pts.)
Proof: $\cup(\beta+1) = \cup_{\gamma \in \beta+1} \gamma = \cup_{\gamma \in \beta\cup\{\beta\}} \gamma = (\cup_{\gamma \in \beta} \gamma) \cup \beta = \beta$ because, since β is an ordinal, for any $\gamma \in \beta$, $\gamma \subseteq \beta$.

4d. *Show that every natural number is an ordinal.* (2 pts.)
Proof: $0 = \emptyset$ and $\emptyset$ is an ordinal. If n is an ordinal $n + 1$ is an ordinal buy 4c.

4e. *Assuming the Axiom of Regularity, show that every ordinal is either $\emptyset$ or contains $\emptyset$ as an element.* (3 pts.)
Proof: Assume $\alpha \neq \emptyset$ is an ordinal. Then by 4a, α has a least element, say β. By 4a again, $\beta \cap \alpha = \emptyset$. But, since $\beta \in \alpha$, $\beta \subseteq \alpha$. Hence $\beta = \beta \cap \alpha = \emptyset$.

4f. *Show that the set ω of natural numbers is an ordinal.* (2 pts.)
Proof: This is clear.

4g. *Assuming the Axiom of Regularity, show that every element of an ordinal is an ordinal.* (3 pts.)
Proof: Let X be an ordinal. Let $x \in X$.
We first show that x is an $\in$-complete set. Let $y \in x$. Let $z \in y$. Then $z \in y \in x$. Hence by 4a, $z \in x$. Thus $y \subseteq x$.
Now we show x is $\in$-connected. Let $y, z \in x$ be two distinct elements of x. Since $x \subseteq X$, y and z are elements of X. Hence either $y \in z$ or $z \in y$.

4h. *An ordinal* $\alpha \neq \varnothing$ *is called a* ***limit ordinal*** *if* α *is not of the form* $\beta + 1$ *for some* $\beta \in \alpha$. *Show that* ω *is a limit ordinal but that no natural number is a limit ordinal*. (5 pts.)

Proof: Suppose $\omega = \beta + 1$ for some $\beta \in \omega$. Then $\omega = \beta+1 \in \omega$ and we know this is

4i. Show that if X is a set of ordinals such that for all $\alpha, \beta \in X$ either $\alpha \subseteq \beta$ or $\alpha = \beta$ or $\beta \subseteq \alpha$, then $\cup X$ is an ordinal. (5 pts.)