Ordinals

Summer Midterm II
15th of June, 1999
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Prelude: Let X be a set and $<$ be a total order on X. We say that $(X, <)$ is a **well-ordered** set (or that $<$ well-orders X) if every nonempty subset of X contains a minimal element for that order, i.e., if for every nonempty subset A of X, there is an $m \in A$ such that $m \leq a$ for all a in A. Clearly, given A, such an m is unique.

Of course, subsets of the well-ordered set X inherit the well-order of X.

If $(X, <)$ is an ordered set and $x \in X$, we define

$$s(x) = \{y \in X : y < x\} \text{ (the initial segment of } x\text{)}$$

If X is a set, we set $X^+ = X \cup \{X\}$. By the axiom of regularity, X is a proper subset of X^+.

1. Assume X is a well-ordered set. Order X^+ by extending the order of X and stating that X is larger than its elements (i.e. put the element X to the very end of X). Show that X^+ is also a well-ordered set. (3 pts.)

2. (Transfinite Induction) Let $(X, <)$ be a well-ordered set and let $A \subseteq X$ be such that for all $x \in X$, if $s(x) \subseteq A$, then $x \in A$. Show that $A = X$. (5 pts.)

3. Let X and Y be two well-ordered sets. Let
$A = (X \times \{0\}) \cup (Y \times \{1\})$.
Order A as follows:

$(x_1, 0) < (x_2, 0)$ for all x_1 and x_2 in X and $x_1 < x_2$.
$(y_1, 1) < (y_2, 1)$ for all y_1 and y_2 in Y and $y_1 < y_2$.
$(x, 0) < (y, 1)$ for all $x \in X$ and $y \in Y$.

Show that the above relation well-orders A. (4 pts.)

An **ordinal** is a well-ordered set α such that $\beta = s(\beta)$ for all $\beta \in \alpha$. Thus an ordinal is a set well-ordered by the relation $\in$:

$$\text{For all } \beta, \gamma \in \alpha,\ \gamma < \beta \text{ iff } \gamma \in \beta.$$

4. Show that $\varnothing$ is an ordinal. (2 pts.)

5. Show that if $\alpha \neq \varnothing$ is an ordinal, then $\varnothing \in \alpha$ and $\varnothing$ is the least element of α. (7 pts.)

6. Show that if α is an ordinal and $\beta \in \alpha$, then $\beta \subset \alpha$. (2 pts.)

7. Show that every element of an ordinal is an ordinal. (2 pts.)

8. Show that if α is an ordinal, then α^+ is also an ordinal. (2 pts.)

9. Let α be an ordinal and $\beta \in \alpha$. Show that either $\beta^+ \in \alpha$ or $\beta^+ = \alpha$. (8 pts.)

10. In exercise 3 take $X = \omega$ and $Y = 1 = \{0\}$. Show that the well-ordered set A obtained there is isomorphic to the ordinal ω^+, i.e. there is an order-preserving bijection from A onto ω^+. (4 pts.)

11. In exercise 3 take $X = 1 = \{0\}$ and $Y = \omega$. Show that the well-ordered set A obtained there is **isomorphic** to ω, i.e. there is an order-preserving bijection from A onto ω. (4 pts.)

12. Let α, β be ordinals. Show that either $\alpha < \beta$ or $\alpha = \beta$ or $\beta < \alpha$. (18 pts.)

13. Show that the union of a set of ordinals is an ordinal. (3 pts.)

14. Let α and β be two ordinals. Let $f : \alpha \to \beta$ be a strictly increasing function. Show that if f is onto, then $\alpha = \beta$ and f is the identity map. (18 pts.)

15. Show that every well-ordered set is isomorphic to an ordinal. (18 pts.)