Quiz on Ordinals

Math 111
April 1999
Ali Nesin

We assume the Axiom of Regularity throughout.

An **ordinal** is a set α such that

i) For any two distinct elements x, y of α, either $x \in y$ or $y \in x$. (Such a set is called **ε-connected**).

ii) Any element of α is a subset of α. (Such a set is called **ε-complete**).

1. Show that if α is an ordinal, then $\cup\alpha \subseteq \alpha$.
2. Show that if α is an ordinal, so is $\alpha + 1$. (Recall that $\alpha + 1$ is defind to be $\alpha \cup \{\alpha\}$).
3. Show that if β is an ordinal, then $\cup(\beta+1) = \beta$.
4. Show that every natural number is an ordinal.
5. Show that every ordinal is either $\emptyset$ or contains $\emptyset$ as an element.
6. Show that the set ω of natural numbers is an ordinal.
7. Show that the membership relation $\in$ totally orders an ordinal.
8. Show that every element of an ordinal is an ordinal.
9. Show that every nonempty subset of an ordinal has a least element.
10. An ordinal α is called a **limit ordinal** if α is not of the form $\beta + 1$ for some $\beta \in \alpha$. Show that ω is a limit ordinal but that no natural number is a limit ordinal.
11. Show that if X is a set of ordinals such that for all $\alpha, \beta \in X$ either $\alpha \subseteq \beta$ or $\alpha = \beta$ or $\beta \subseteq \alpha$, then $\cup X$ is an ordinal.
12. **Transfinite Induction.** Let $\varphi(x)$ be a first-order statement. Assume that i) $\varphi(\emptyset)$ holds, ii) If $\varphi(\alpha)$ holds, then $\varphi(\alpha+1)$ holds also, iii) If $\varphi(\beta)$ holds for all elements β of a limit ordinal α, then $\varphi(\alpha)$ holds. Show that $\varphi(\alpha)$ holds for all ordinals α.
13. Let α be an ordinal and $X \subseteq \alpha$. Show that the set $\{y \in \alpha: y \in x \text{ for some } x \in X\}$ is also an ordinal.