Quiz on Ordinals
Math 111
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We assume the Axiom of Regularity throughout.

An ordinal is a set $\alpha$ such that

i) For any two distinct elements $x, y$ of $\alpha$, either $x \in y$ or $y \in x$. (Such a set is called $\varepsilon$-connected).

ii) Any element of $\alpha$ is a subset of $\alpha$. (Such a set is called $\varepsilon$-complete).

1. Show that if $\alpha$ is an ordinal, then $\cup \alpha \subseteq \alpha$.
2. Show that if $\alpha$ is an ordinal, so is $\alpha + 1$. (Recall that $\alpha + 1$ is defined to be $\alpha \cup \{\alpha\}$).
3. Show that if $\beta$ is an ordinal, then $\cup (\beta + 1) = \beta$.
4. Show that every natural number is an ordinal.
5. Show that every ordinal is either $\emptyset$ or contains $\emptyset$ as an element.
6. Show that the set $\omega$ of natural numbers is an ordinal.
7. Show that the membership relation $\in$ totally orders an ordinal.
8. Show that every element of an ordinal is an ordinal.
9. Show that every nonempty subset of an ordinal has a least element.
10. An ordinal $\alpha$ is called a limit ordinal if $\alpha$ is not of the form $\beta + 1$ for some $\beta \in \alpha$. Show that $\omega$ is a limit ordinal but that no natural number is a limit ordinal.
11. Show that if $X$ is a set of ordinals such that for all $\alpha, \beta \in X$ either $\alpha \subseteq \beta$ or $\alpha = \beta$ or $\beta \subseteq \alpha$, then $\cup X$ is an ordinal.
12. Transfinite Induction. Let $\varphi(x)$ be a first-order statement. Assume that i) $\varphi(\emptyset)$ holds, ii) If $\varphi(\alpha)$ holds, then $\varphi(\alpha + 1)$ holds also, iii) If $\varphi(\beta)$ holds for all elements $\beta$ of a limit ordinal $\alpha$, then $\varphi(\alpha)$ holds. Show that $\varphi(\alpha)$ holds for all ordinals $\alpha$.
13. Let $\alpha$ be an ordinal and $X \subseteq \alpha$. Show that the set $\{y \in \alpha: y \in x$ for some $x \in X\}$ is also an ordinal.