**Math 131**
**Make Up Exam**
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Show your work. Bare answers will not be accepted, not even for partial credit.
Passing grade is 50.

1. Find the remainder when $37^{126}$ is divided by 13. (5 pts.)

2. Show that $\sum_{i=0}^{n} (-2)^i \binom{n}{i} = (-1)^n$. (5 pts.)

3. Let $d$ be the greatest common divisor of the two positive integers $a$ and $b$.

3a. Show that there are integers $x$ and $y$ such that $ax + by = d$. (10 pts.)

3b. Let $a = 23023$, $b = 24871$. Find $d$, $x$ and $y$ as above. (10 pts.)

4. Let $aX^2 + bX + c \in \mathbb{Z}[X]$ have two distinct integer roots. Show that $a$ must divide both $b$ and $c$. (10 pts.)

5. Let $b, c \in \mathbb{Z}$. Show that the necessary and sufficient condition for the equation $x^2 + bx + c = 0$ to have a root in $\mathbb{Z}$ is that $b^2 - 4c$ is a perfect square in $\mathbb{Z}$. (10 pts.)

6. Let $f(X) \in \mathbb{Z}[X]$ be a monic polynomial (i.e. the leading coefficient of $f$ is 1). Show that all the rational roots of $f$ are integers. (10 pts.)

7. Let $f(X) = a_nX^n + a_{n-1}X^{n-1} + ... + a_0$ be a real polynomial with $a_n \neq 0$.

7a. Let $\alpha$ be a real root of $f$. Show that $|\alpha| \leq \sup\{1, |a_{n-1}/a_n| + ... + |a_0/a_n|\}$. (10 pts.)

7b. Deduce that there is an algorithm for finding all the integer roots of a polynomial in $\mathbb{Z}[X]$. (5 pts.)

8. Let $f(X) = a_nX^n + a_{n-1}X^{n-1} + ... + a_0 \in \mathbb{Z}[X]$ be a polynomial with $a_n \neq 0$. Let $\alpha$ be a rational root of $f$. Write $\alpha = r/s$ with $r, s \in \mathbb{Z}$ and $\gcd(r, s) = 1$.

8b. Show that $s$ divides $a_n$. (10 pts.)

8c. Using #7a show that $|r| \leq \sup(|a_n|, |a_{n-1}| + ... + |a_0|)$. (10 pts.)

8d. Deduce that there is an algorithm for finding all the rational roots of a polynomial in $\mathbb{Z}[X]$. (5 pts.)