Math 112

Set Theory II
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I. Automorphisms of ($\mathbb{Q}$, <)

a) Let $a, b \in \mathbb{Q}$, $a > 0$. Let $f_{a,b} : \mathbb{Q} \to \mathbb{Q}$ be defined by $f_{a,b}(x) = ax + b$. Show that $f_{a,b}$ is an order preserving bijection.

b) Let $a, b \in \mathbb{Q}$, with $a < b$ and $k \in \mathbb{Z}$. Show that there is an order preserving bijection between the rational intervals $[k, k + 1]$ and $[a, b]$.

c) Show that there are uncountably many order preserving bijections of $\mathbb{Q}$.

II. Automorphisms of ($\mathbb{R}$, +, ×)

a) Show that any additive map f from $\mathbb{Z}$ into $\mathbb{R}$ is given by $f(x) = rx$ for some $r \in \mathbb{R}$.

b) Show that any additive map f from $\mathbb{Q}$ into $\mathbb{R}$ is given by $f(x) = rx$ for some $r \in \mathbb{R}$.

c) Show that the only additive and multiplicative map f from $\mathbb{Q}$ into $\mathbb{Q}$ is $\mathrm{Id}_{\mathbb{Q}}$.

d) Show that the only additive and multiplicative map f from $\mathbb{R}$ into $\mathbb{R}$ is $\mathrm{Id}_{\mathbb{R}}$.

III. For $i = 1, 2$, let $(\mathbb{R}_i, +_i \times_i, 0_i, 1_i)$ be two structures satisfying the axioms of real numbers (ordered complete field). Show that there is a unique bijection $f : \mathbb{R}_1 \to \mathbb{R}_2$ such that

i. $f(x +_1 y) = f(x) +_2 f(y)$
ii. $f(x \times_1 y) = f(x) \times_2 f(y)$.