Math 111 / Math 113 Set Theory
Midterm
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Ali Nesin

Definitions:
$0 = \varnothing$.
For a set x, $S(x) = x \cup \{x\}$.
A set X is ***inductive*** if it contains 0 and if for all $x \in X$, $S(x)$ is also an element of X.
ω is the smallest inductive set, i.e. it is the intersection of all inductive sets.

1. *Show that if x and y are sets then so is* $\{\{x\}, \{x, y\}\}$. (7 pts.)

Proof: If x and y are sets, then there is an axiom that states that $\{x, y\}$ is a set. Taking $x = y$, we see that $\{x\}$ is a set as well. By the same axiom $\{\{x\}, \{x, y\}\}$ is a set.

2. *For two sets x and y, define the* ***pair*** (x, y) *to be the set* $\{\{x\}, \{x, y\}\}$. *Show that for any four sets* x, y, z, t, $(x, y) = (z, t)$ *if and only if* $x = z$ *and* $y = t$. (7 pts.)

Proof: It is clear that if $x = z$ and $y = t$ then $(x, y) = (z, t)$.
Conversely, suppose that $(x, y) = (z, t)$. By definition, this means that

$$\{\{x\}, \{x, y\}\} = \{\{z\}, \{z, t\}\}.$$

Therefore the set $\{x\}$ which is an element of the set $\{\{x\}, \{x, y\}\}$ is also an element of the set $\{\{z\}, \{z, t\}\}$. Hence either $\{x\} = \{z\}$ or $\{x\} = \{z, t\}$. In the first case $x = z$ and in the second case $z = t = x$. Thus in both cases $x = z$. It remains to show that $y = t$. Since

$$\{\{x\}, \{x, y\}\} = \{\{z\}, \{z, t\}\}$$

and since $\{x\} = \{z\}$, we must have $\{x, y\} = \{z, t\}$. Then the equality $x = z$ forces the equality $y = t$.

3. *Let X and Y be two set. Let* $Z = \wp(\wp(X \cup Y))$. *Show that* $(x, y) \in Z$ *for all* $x \in X$ *and* $y \in Y$. (7 pts.)

Proof: Note that $\wp(\wp(X \cup Y))$ is a set by two of the axioms of set theory. Since $x \in X$ and $X \subseteq X \cup Y$, we have $x \in X \cup Y$. Similarly $y \in X \cup Y$. It follows that

$$\{x\} \subseteq X \cup Y \text{ and } \{x, y\} \subseteq X \cup Y.$$

Hence

$$\{x\} \in \wp(X \cup Y) \text{ and } \{x, y\} \in \wp(X \cup Y).$$

Therefore,

$$\{\{x\}, \{x, y\}\} \subseteq \wp(X \cup Y).$$

This gives

$$\{\{x\}, \{x, y\}\} \in \wp(\wp(X \cup Y)) = Z.$$

4. *Show that the collection of all pairs* (x, y) *for* $x \in X$ *and* $y \in Y$ *is a set. We denote this set by* $X \times Y$. (7 pts.)

Proof: This is the collection $\{(x, y) \in \wp(\wp(X \cup Y)) : x \in X, y \in Y\}$. To show that this is a set we will use the third axiom of set theory given in class, namely that if Z is a set and $\varphi(z)$ is a formula, then the collection $\{z \in Z : \varphi(z)\}$ is a set.
Let $\alpha(x, u)$ be $x \in u \wedge \forall t\ (t \in u \to t = x)$. Then $\alpha(x, u)$ holds if and only if $u = \{x\}$.
Let $\beta(x, y, v)$ be $x \in v \wedge y \in v \wedge \forall t\ (t \in u \to (t = x \vee t = y))$. Then $\beta(x, y, v)$ holds if and only if $v = \{x, y\}$.

Let $\gamma(x, y, z)$ be $\exists u\, \exists v\, (\alpha(x, u) \wedge \beta(x, y, v) \wedge \beta(u, v, z))$. Then $\gamma(x, y, z)$ holds if and only if $z = \{\{x\}, \{x, y\}\} = (x, y)$.
Let $\varphi(z)$ be $\exists x\, \exists y\, (x \in X \wedge y \in Y \wedge \gamma(x, y, z))$. Then $\varphi(z)$ holds if and only if $z = (x, y)$ for some $x \in X$ and $y \in Y$.
Thus the collection $\{(x, y) \in \wp(\wp(X \cup Y)) : x \in X, y \in Y\}$ can also be expressed as

$$\{z \in \wp(\wp(X \cup Y)) : \varphi(z)\}.$$

Therefore by the axiom stated above (axiom of definable subsets or the axion of extensionality), this collection, i.e. $X \times Y$, is a set.

5. *Show that the collection of all pairs* (x, y) *such that* $y = S(x)$ *for some* $x \in \omega$ *is a subset of* $\omega \times \omega$. (7 pts.)

Proof: We only need to show that the collection $\{(x, y) \in \omega \times \omega : y = S(x)\}$ is a set. Since we know that $\omega \times \omega$ is a set, we only need to express the condition $y = S(x)$ as a formula $\varphi(z)$.
Let $\varepsilon(x, y, z)$ be $\forall t\, (t \in z \leftrightarrow t \in x \vee t \in y)$. Then $\varepsilon(x, y, z)$ holds if and only if $z = x \cup y$.
Let $\psi(x, y)$ be $\exists t\, (\alpha(x, t) \wedge \varepsilon(x, t, y))$. (Here the formula α is as in the previous question). Then $\psi(x, y)$ holds if and only if $y = x \cup \{x\} = S(x)$.
Thus the collection $\{(x, y) \in \omega \times \omega : y = S(x)\}$ is also the collection

$$\{z \in \omega \times \omega : \exists x\, \exists y\, (\gamma(x, y, z) \wedge \psi(x, y)\},$$

(here the formula γ is as in the previous question) and hence is a set by the Axiom of definable sets (the famous Axiom 3).

6. *Show that for all* $n, m \in \omega$, *if* $n \in m$ *then* $n \subseteq m$. (7 pts.)

Proof: We proceed by induction on m. If $m = 0$, then the statement is vacuously true. Assume the statement holds for m. Let $n \in S(m) = m \cup \{m\}$. Then either $n \in m$ or $n \in \{m\}$. In the first case by induction we have $n \subseteq m$; since $m \subseteq m \cup \{m\} = S(m)$, in that case we get $n \subseteq S(m)$. In the second case we nmust have $m = n$ and again $n = m \subseteq m \cup \{m\} = S(m)$.

7. *Show that for all* $n, m \in \omega$, *if* $S(n) = S(m)$ *then either* $n \in m$ *or* $n = m$. (7 pts.)

Proof: Assume $S(n) = S(m)$. Then, by definition, $n \cup \{n\} = m \cup \{m\}$. Since n is an element of the set $n \cup \{n\}$, this implies that $n \in m \cup \{m\}$. Thus either $n \in m$ or $n \in \{m\}$. In the second case we get $n = m$.

8. *Show that* $S : \omega \to \omega$ *is a one-to-one function.* (7 pts.)

Proof: Assume that for $n, m \in \omega$, $S(n) = S(m)$ but that $n \neq m$. By question 7, either $n \in m$ or $n = m$. Therefore $n \in m$. By question 6, $n \subseteq m$. By symmetry $m \subseteq n$. Hence $n = m$.

9. *Show that* $S(\omega) = \omega \setminus \{0\}$. (7 pts. **Note:** Here $S(\omega)$ denotes the image of ω under the function ω and is not $\omega \cup \{\omega\}$.)

Proof: Since $S(n) = n \cup \{n\}$, $S(n)$ can never be empty, i.e. $S(n) \neq 0$ and so $S(\omega) \subseteq \omega \setminus \{0\}$. Conversely, we will show that for any $n \in \omega$, either $n = 0$ or $n = S(m)$ for some $m \in \omega$. We proceed by induction. If $n = 0$ the statements holds trivially. Suppose the statement holds for n (we really will not care that the statement holds for n) and show it for $S(n)$. Thus we have to show that $S(n)$ is the S-image of some m. But $S(n)$ is of course S of something, namely of n...

10. *Show that for any* $n \in \omega$, $n \notin n$. (7 pts.)

Proof: We proceed by induction on n. If $n = 0$, then $n = \emptyset$ and of course $n \notin n$. Assume that $n \notin n$ and show that $S(n) \notin S(n)$. Assume to the contrary that $S(n) \in S(n) = n \cup \{n\}$. Then either $S(n) \in n$ or $S(n) = n$. Bu question 6, $S(n) \subseteq n$ in both cases. But since, $n \in n \cup \{n\} = S(n)$, this implies that $n \in n$, a contradiction.

11. *For* $n, m \in \omega$ *define the binary relation* $n < m$ *by* $n \in m$. *Show that this relation is an order on* ω. (7 pts.)

Proof: We need to show that for all $n, m, k \in \omega$, we have

a) $n \notin n$

and

b) if $n \in m$ and $m \in k$ then $n \in k$.

The first one is given bu question 10. Assume now $n \in m \in k$. By question 6, $n \in m \subseteq k$. Hence $n \in k$.

12. *Show that the order* $<$ *is a total order*. (7 pts.)

Proof: We first need a lemma.

Lemma: *For all* $n, m \in \omega$, *if* $n \in m$ *then either* $S(n) \in m$ or $S(n) = m$.

Proof: We proceed by induction on m. If $m = 0$ there is nothing to prove. Assume now that m is given so that the statement

For all $n \in \omega$, *if* $n \in m$ *then either* $S(n) \in m$ or $S(n) = m$.

holds (the inductive hyothesis). We will show that

For all $n \in \omega$, *if* $n \in S(m)$ *then either* $S(n) \in S(m)$ or $S(n) = S(m)$.

Let $n \in S(m)$ be any. We will show that either $S(n) \in S(m)$ or $S(n) = S(m)$. Since $n \in S(m) = m \cup \{m\}$, either $n \in m$ or $n = m$. In the second case $S(n) = S(m)$. In the first case, by induction, either $S(n) \in m$ or $S(n) = m$ and in both cases $S(n) \in S(m)$. This proves the lemma.

Now we show that

for all $m \in \omega$, either $n \in m$ or $n = m$ or $m \in n$

by induction on n. Assume first $n = 0$ and choose an $m \in \omega$. Since $m \in 0$ is impossible we need to show that

either $0 = m$ or $0 \in m$.

We do this by induction on m. If $m = 0$ there is nothing to prove. Assume that this holds for m, we prove it for $S(m)$. If $m = 0$, $0 = m \in S(m)$. If $m \neq 0$, then $0 \in m \subseteq S(m)$. Thus the statement is proved for $n = 0$.

Assume now that

for any m, either $n \in m$ or $n = m$ or $m \in n$.

We will show that the same statement holds for $S(n)$ instead of n, namely that

for any m, either $S(n) \in m$ or $S(n) = m$ or $m \in S(n)$.

Let $m \in \omega$ be any. By induction we have three possibilities

$n \in m$ or $n = m$ or $m \in n$.

In the second case, $m = n \in S(n)$ and we are done.

In the third case, $m \in n \subseteq S(n)$ and we are done again.

We are left with the first case $n \in m$. But this case is dealt by the lemma above.

13. *Show that any nonempty subset of* ω *has a least element for this order*. (8 pts.)

Proof: Let X be a nonempty subset of ω. Assume that X does not have a least element. We will first show by induction on n that

$$\text{for all } m < n,\ m \notin X.$$

If $n = 0$ this holds trivially. Assume that the statement holds for n. Let $m < S(n)$. Thus $m \in S(n) = n \cup \{n\}$ and either $m \in n$ or $m = n$. In the first case $m < n$ and by induction m cannot be an element of X. In the second case, if n were an element of X, then n would be the least element of X because of the inductive hypothesis; thus $n \notin X$ either. Therefore the statement is proved. Now we show that $X = \emptyset$. Assume $n \in X$. Then $n < S(n)$ and the statement which has just been proven is false for $S(n)$, a contradiction. Therefore $X = \emptyset$.

14. *Show that for any nonempty subset* X *of* ω *there is an element* $x \in X$ *such that* $x \cap X = \emptyset$. (8 pts.)

Proof: Let x be the least element of X. (It exists by question 13). If $y \in x \cap X$, then y would be an element of X which is smaller than x, a contradiction. Thus $x \cap X = \emptyset$.