Math 111 Resit

Ali Nesin

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1a. Find a nonempty set A such that $A \cap \wp(A) = \emptyset$.

1b. Find a set A such that $A \cap \wp(A) \neq \emptyset$.

1c. Is there a set A such that $\wp(A) \cap \wp(\wp(A)) = \emptyset$?

2a. Let A be a set and let $B = \{x \in A : x \notin x\}$. Without using the axiom of regularity[1], show that $B \notin A$. Conclude that $B \notin B$.

3. Let a and b be two integers.

3a. Show that $\gcd(a, b) = \gcd(a - b, b)$.

3b. Let d be the greatest common divisor of a and b. Show that there are integers x and y such that $xa + yb = d$.

4. Let $a < b$ be two fixed real numbers. Find (explicitely) a bijection $f_{a,b}$ between the open intervals $(0, 1)$ and (a, b). What is its inverse? What is $f_{a,b} \circ f_{c,d}^{-1}$?

5. A set X is called **complete** if every element of X is a subset of X.

5a. Give infinitely many examples of complete sets.

5b. Show that if A is a set of complete sets, then $\cap A$ and $\cup A$ are also complete.

5c. Show that if X is complete, then $X \cup \{X\}$ is also complete.

6. A set X is $\in$**-connected** if for any two distinct elements x, y of X, either $x \in y$ or $y \in x$.

6a. Give infinitely many examples of $\in$-connected sets.

6b. Show that a subset of an $\in$-connected set is $\in$-connected.

6c. Show that if X is $\in$-connected, then $X \cup \{X\}$ is also $\in$-connected.

7. Let $(a_i)_{i \in \mathbb{N}}$ be an increasing and $(b_i)_{i \in \mathbb{N}}$ a decreasing sequence of real numbers with $a_n < b_m$ for all $n, m \in \mathbb{N}$. Show that $\bigcap_{i \in N} [a_i, b_i] \neq \emptyset$. Show that this result is false for the set $\mathbb{Q}$ of rational numbers.

8. Can you find an increasing sequence $(a_i)_{i \in \mathbb{N}}$ and a decreasing sequence $(b_i)_{i \in \mathbb{N}}$ of real numbers with $a_n < b_m$ for all $n, m \in \mathbb{N}$ such that $\bigcap_{i \in N} [a_i, b_i) = \emptyset$?

9. Let a and b be two real numbers. Show that if the closed interval $[a, b]$ is covered by a set of open intervals, then only finitely many of these intervals is enough to cover $[a, b]$.

10. Let $x \in \mathbb{R}$. Show that $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges.

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[1] If you do not know the axiom of regularity, just ignore it.

11. [Cantor-Schröder-Bernstein] Let A be a set and A' a subset of A. Assume that there is a bijection $f\colon A \to A'$ betweeen A and A'. Let B be any set such that $A' \subseteq B \subseteq A$. The purpose of this exercise is to show that there is a bijection between B and A.

Let $Q = B \setminus A'$.

Let $\Gamma = \{X \subseteq A : Q \cup f(X) \subseteq X\}$.

Let $T = \cap\, \Gamma = \bigcap_{X \in \Gamma} X$.

11a. Show that $T \in \Gamma$.

11b. Show that $Q \cup f(T) \in \Gamma$.

11c. Show that $T = Q \cup f(T)$. (Hint: Use a and b).

11d. Show that $B = T \cup (A' \setminus f(T))$. (Hint: Use c).

11e. Show that $T \cap (A' \setminus f(T)) = \varnothing$.

11f. Show that there is a bijection between B and A. (Hint: Use parts d and e).