Math 111

Midterm 1 Correction; November 1997

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1. For two natural numbers n and m, we say that m divides n if there is a natural number x such that $n = mx$. We denote this by $m \mid n$.

1a. Show that $n \mid n$.

1b. Show that if $p \mid m$ and $m \mid n$, then $p \mid n$.

1c. Show that if $n \mid m$ and $m \mid n$ then $n = m$.

Correction of 1a. We need to find an x such that $n = nx$. Since $n = n1$, we can take $x = 1$.

1b. Since $p \mid m$ and $m \mid n$, there are x and y such that $m = px$ and $n = my$. We need to find a z such that $n = pz$. We have,

$$n = my = (px)y = p(xy).$$

Therefore it is enough to take $z = xy$.

1c. Since $n \mid m$ and $m \mid n$, there are x and y such that $m = nx$ and $n = my$. we have,

$$n = my = (nx)y = n(xy).$$

If $n \neq 0$, by simplifying, we get $xy = 0$, hence $x = 1$ and $m = nx = n1 = n$. If $n = 0$, $m = nx = 0x = 0 = n$, again the equality.

2. A natural number p is called **prime** if $p \neq 1$ and p is divisible only by 1 and p.

Show that any natural number $\neq 1$ is divisible by a prime. (**Hint:** Let a be a natural number. Define $X_a = \{x \in \mathbf{N} : x \mid a\}$. Show that the smallest element of X_a (why does it exist?) is prime).

3. Let X and Y be two sets and $f: X \to Y$ a function. For any two subsets A and B of X and any two subsets C and D of Y show the following:

3a. $f(A \cup B) = f(A) \cup f(B)$.

3b. $f(A \cap B) \subseteq f(A) \cap f(B)$.

3c. Give an example where the equality in 3b does not hold.

3d. $f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$.

3e. $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$.

4. Show that for any positive natural number,

$$\frac{1}{1}\frac{1}{3} + \frac{1}{3}\frac{1}{5} + \frac{1}{5}\frac{1}{7} + \ldots + \frac{1}{2n-1}\frac{1}{2n+1} = \frac{n}{2n+1}$$

5. Let A be a nonempty subset of $\mathbb{Z}$ satisfying the following property:

For all a and b of A, $a - b$ is also in A. (*)

5a. Show that $0 \in A$.

5b. Show that if $a \in A$, then $-a \in A$.

5c. Show that if a and b are in A, then $a + b \in A$.

5d. Give 5 examples of nonempty subsets A of $\mathbb{Z}$ that satisfies (*).

5e*. Find all the nonempty subsets of $\mathbb{Z}$ that satisfies (*).