1. Let $X$ be a set and $R$ be a binary relation on $X$. Show that there is a least equivalence relation $\equiv$ on $X$ such that for all $x, y \in X$ if $x R y$ then $x \equiv y$.

2. Let $(X_n)_n$ be a sequence of subsets of a set $X$. We define $\liminf X_n$ and $\limsup X_n$ as follows:

   An element $a$ of $X$ belongs to $\liminf X_n$ if and only if there exists a natural number $n_0$ such that $a$ is in $X_n$ for all $n > n_0$.

   An element $a$ of $X$ belongs to $\limsup X_n$ if and only if for every natural number $n_0$ there exists an index $n > n_0$ such that $a$ is in $X_n$.

   2i. Show that $\limsup X_n$ consists of those elements which are in $X_n$ for infinitely many $n$, while $\liminf X_n$ consists of those elements which are in $X_n$ for all but finitely many $n$.

   2ii. Show that $\liminf X_n = \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} X_m$ and that $\limsup X_n = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} X_m$.

   2iii. Let $X_n = \{ n, n+1, \ldots, 2n \}$. Find $\liminf X_n$ and $\limsup X_n$.

   2iv. Assume $X_{n+1} \subseteq X_n$ for all $n$. Find $\liminf X_n$ and $\limsup X_n$.

   2v. Find an example where $\liminf X_n \neq \limsup X_n$.

3. Let $R$ be a commutative ring with 1. Let $\leq$ be a total order on $R$ such that for all $x, y, z \in R$,

   a) if $x \leq y$ then $x + z \leq y + z$.
   b) if $0 < x$ and $0 < y$ then $0 < xy$.

   Show that for all $x, y, z \in R$

   3i. If $x < y$ then $-y < -x$.
   3ii. If $x < y$ then $x + z < y + z$.
   3iii. If $x \leq y$ and $0 \leq z$ then then $xz \leq yz$.
   3iv. If $x \leq y$ and $0 \geq z$ then $xz \geq yz$.
   3v. $-1 < 0 < 1$.
   3vi. If $x \neq 0$ and $y \neq 0$ then $xy \neq 0$.
   3vii. $x^2 \geq 0$.
   3viii. If $x < 0$ then $x$ is not a sum of squares in $R$.
   3ix. $-1$ is not a sum of squares in $R$.

   3x. If $x \geq 0$ and $x$ has a multiplicative inverse in $R$ then $x^{-1} > 0$.

   Define $|x|$ as follows: $|x| = x$ if $x \geq 0$ and $|x| = -x$ if $x \leq 0$.

   3xi. Show that $|x| \geq 0$.
   3xii. Show that $|xy| = |x| |y|$.
   3xiii. Show that $|x + y| \leq |x| + |y|$.
   3xiv. Show that $|x - y| \geq |x| - |y|$.

   Define $d(x, y) = |x - y|$.

   3xv. Show that $d(x, y) = 0$ if and only if $x = y$.
   3xvi. Show that $d(x, y) = d(y, x)$. 
3xvii. Show that \( d(x, y) \leq d(x, z) + d(z, y) \).
The last three properties show that \( d \) is a metric on \( R \).

3xviii. Show that the maps \( R \times R \rightarrow R \) defined by \( (x, y) \mapsto x - y \) and \( (x, y) \mapsto xy \) and