Math 111
Final
2006 Fall

1. Let X be a set and R be a binary relation on X. Show that there is a least equivalence relation $\equiv$ on X such that for all $x, y \in X$ if xRy then $x \equiv y$.

2. Let $(X_n)_n$ be a sequence of subsets of a set X. We define liminf X_n and limsup X_n as follows:

An element a of X belongs to liminf X_n if and only if there exists a natural number n_0 such that a is in X_n for all $n > n_0$.

An element a of X belongs to limsup X_n if and only if for every natural number n_0 there exists an index $n > n_0$ such that a is in X_n.

2i. Show that limsup X_n consists of those elements which are in X_n for infinitely many n, while liminf X_n consists of those elements which are in X_n for all but finitely many n.

2ii. Show that

$$\liminf X_n = \bigcup_{n=1}^{\infty}\left(\bigcap_{m=n}^{\infty} X_m\right)$$

and that

$$\limsup X_n = \bigcap_{n=1}^{\infty}\left(\bigcup_{m=n}^{\infty} X_m\right).$$

2iii. Let $X_n = \{n, n+1, .., 2n\}$. Find liminf X_n and limsup X_n.

2iv. Assume $X_{n+1} \subseteq X_n$ for all n. Find liminf X_n and limsup X_n.

2v. Find an example where liminf $X_n \neq$ limsup X_n.

3. Let R be a commutative ring with 1. Let $\leq$ be a total order on R such that for all $x, y, z \in R$,

a) if $x \leq y$ then $x + z \leq y + z$.

b) if $0 < x$ and $0 < y$ then $0 < xy$.

Show that for all $x, y, z \in R$

3i. If $x < y$ then $-y < -x$.

3ii. If $x < y$ then $x + z < y + z$.

3iii. If $x \leq y$ and $0 \leq z$ then then $xz \leq yz$.

3iv. If $x \leq y$ and $0 \geq z$ then then $xz \geq yz$.

3v. $-1 < 0 < 1$.

3vi. If $x \neq 0$ and $y \neq 0$ then $xy \neq 0$.

3vii. $x^2 \geq 0$.

3viii. If $x < 0$ then x is not a sum of squares in R.

3ix. -1 is not a sum of squares in R.

3x. If $x \geq 0$ and x has a multiplicative inverse in R then $x^{-1} > 0$.

Define $|x|$ as follows: $|x| = x$ if $x \geq 0$ and $|x| = -x$ if $x \leq 0$.

3xi. Show that $|x| \geq 0$.

3xii. Show that $|xy| = |x|\,|y|$.

3xiii. Show that $|x + y| \leq |x| + |y|$.

3xiv. Show that $|x - y| \geq \big|\,|x| - |y|\,\big|$.

Define $d(x, y) = |x - y|$.

3xv. Show that $d(x, y) = 0$ if and only if $x = y$.

3xvi. Show that $d(x, y) = d(y, x)$.

3xvii. Show that $d(x, y) \leq d(x, z) + d(z, y)$.
The last three properties show that d is a metric on R.
3xviii. Show that the maps $R \times R \to R$ defined by $(x, y) \mapsto x - y$ and $(x, y) \mapsto xy$ and