Set Theory

December, 2005
Ali Nesin
(50 minutes)

I. Let $(X_n)_n$ be a sequence of subsets of a set X. We define liminf X_n and limsup X_n as follows:

An element a of X belongs to liminf X_n if and only if there exists a natural number n_0 such that a is in X_n for all $n > n_0$.

An element a of X belongs to limsup X_n if and only if for every natural number n_0 there exists an index $n > n_0$ such that a is in X_n.

Ii. Show that limsup X_n consists of those elements which are in X_n for infinitely many n, while liminf X_n consists of those elements which are in X_n for all but finitely many n.

Iii. Show that

$$\liminf X_n = \bigcup_{n=1}^{\infty}\left(\bigcap_{m=n}^{\infty} X_m\right)$$

and that

$$\limsup X_n = \bigcap_{n=1}^{\infty}\left(\bigcup_{m=n}^{\infty} X_m\right).$$

II. Explain the following "paradox":

Elif is in an empty room with a window to the garden. Murat is outside the room, just in front of the door with infinitely many small balls numbered by natural numbers in his pocket: 0, 1, 2, 3, and so forth.

For some unknown reason, Murat throws the balls number 0 and 1 to Elif. Elif, not knowing how to react to such a strange behaviour, throws the ball number 0 from the window.

Half a second second later, Murat, in his turn, not being able to explain Elif's motifs to throw the ball number 0 to the garden, decides to repeat the experience: He sends the balls number 2 and 3 to Elif and waits for Elif's move. Elif, this time annoyed, throws the ball number 1 to the garden.

Then, 1/4-th of a second later, Murat sends the balls number 4 and 5 to Elif. Elif, without hesitating sends the ball number 2 to the garden.

1/8-th of a second later Murat sends the balls number 6 and 7 to Elif, and Elif throws the ball number 3 to the garden.

$1/2^n$ – th second later, Murat sends the balls number $2n$ and $2n + 1$ to Elif and Elif throws the ball number n to the garden.

Murat thinks as follows: Each time I throw two balls into the room and she throws one ball to the garden. Therefore after each move, the number of balls in the room increases by 1. Therefore, at the end (i.e. after 1 second), there will be infinitely many balls in the room.

Elif thinks as follows: At the n^{th} move I will throw the ball number n from the window. So all the balls will be thrown to the garden and no ball will remain in the room.

Who is right?