Math 112

Final Exam
June 2007
Ali Nesin

Let R be a commutative ring with 1. A subset I of R is called an ***ideal*** of R if $I \neq \emptyset$, $I - I \subseteq I$ and $RI \subseteq I$. In this case we write $I \lhd R$.

1. Show that a subset I of R is an ideal if and only if $0 \in R$, $I + I \subseteq I$ and $RI \subseteq I$. (4 pts.)

2. Given $x \in R$, show that $Rx \lhd R$. (2 pts.)

3. Given $x, y \in R$, show that $Rx + Ry \lhd R$. (2 pts.)

4. Let R be a ring and $\Pi_{\mathbb{N}} R$ be the set of sequences whose terms are in R. Then $\Pi_{\mathbb{N}} R$ is a ring with the usual operations of addition and multiplication of sequences. Let $\oplus_{\mathbb{N}} R$ be the set of sequences with only finitely many non-zero terms. Show that $\oplus_{\mathbb{N}} R \lhd \Pi_{\mathbb{N}} R$. (3 pts.)

5. Show that any ring has at least two ideals. (2 pts.)

6. Let R and S be two rings. Then $R \times S$ is also a ring. Show that an ideal of $R \times S$ is of the form $I \times J$ for some ideals $I \lhd R$ and $J \lhd S$. (6 pts.)

7. Show that an ideal that contains an invertible element must be equal to R. (4 pts.)

8. Show that a ring has only two ideals if and only if it is a field. (5 pts.)

9. Show that the intersection of any set of ideals is an ideal. (4 pts.)

10. Given any subset X of R show that there is a smallest ideal containing X and show that this ideal is $\{r_1x_1 + ... + r_nx_n : n \in \mathbb{N}, r_i \in R, x_i \in X\}$. This is called the ideal ***generated by*** X and is denoted by $\langle X \rangle$. (6 pts.)

11. Let I and J be two ideals of R. Show that $I \cup J$ is an ideal if and only if one of I and J is a subset of the other. (4 pts.)

12. Let $I, J \lhd R$. Show that $I + J \lhd R$ and it is the smallest ideal containing I and J. (3 pts.)

13. Let $e \in R$ be such that $e^2 = e$. Let $f = 1 - e$. Show that $f^2 = f$. (2 pts.) Show that $Re \cap Rf = \{0\}$ and that $R = Re + Rf$. (5 pts.) Show that Re is a ring with an identity (but usually it is not a subring of R; 5 pts.) Show that the map i from $Re \times Rf$ into R defined by

$$i(x, y) = x + y$$

is a bijection that respects addition and multiplication. (5 pts.)

14. Let $I, J \lhd R$. Show that the set

$$IJ = \{x_1y_1 + ... + x_ny_n : n \in \mathbb{N}, x_i \in I, y_i \in J\}$$

is an ideal contained in $I \cap J$. (3 pts.)

15. Let $I, J, K \lhd R$. Show that $I(JK)$ and $(IJ)K$. (3 pts.)

16. Let $I, J, K \lhd R$. What is the relationship between $I(J + K)$ and $IJ + IK$? (4 pts.)

17. Find all ideals of the ring $\mathbb{Z}$. (5 pts.)

18. For $I \lhd R$, let $\sqrt{I} = \{r \in R : r^n \in I \text{ for some } n \in \mathbb{N}\}$. **a)** Show that $\sqrt{I}$ is an ideal of R containing I. (5 pts.) **b)** Show that $\sqrt{\sqrt{I}} = \sqrt{I}$. (3 pts.) **c)** What is the relationship between $\sqrt{(IJ)}$ and $\sqrt{I}\sqrt{J}$? (2 pts.) **d)** Let $I = \{0\}$. Find $\sqrt{I}$ in cases $R = \mathbb{Z}$, $\mathbb{Z}/6\mathbb{Z}$, $\mathbb{Z}/9\mathbb{Z}$, $\mathbb{Z}/18\mathbb{Z}$, $\mathbb{Z}/7000\mathbb{Z}$. (6 pts.) **e)** Find $\sqrt{I}$ in cases $R = \mathbb{Z}$ and $\mathbb{Z} = 9\mathbb{Z}$, $12\mathbb{Z}$, $7000\mathbb{Z}$. (6 pts.)

19. A ***maximal ideal*** of R is by definition an ideal I of R such that $I \neq R$ and if $I \subseteq J \lhd R$ then either $J = I$ or $J = R$. Find all maximal ideals of $\mathbb{Z}$. (4 pts.)

20. Suppose that $R \setminus R^* \lhd R$. Show that $R \setminus R^*$ is the **unique** maximal ideal of R. (3 pts.)

21. Let $I \lhd R$ and $a, b \in R$. Show that either $a + I = b + I$ or $(a + I) \cap (b + I) = \emptyset$. (6 pts.)

22. Let $I \lhd R$. Show that the binary relation $\equiv$ defined by $a \equiv b$ iff $a - b \in I$ is an equivalence relation. (3 pts.) Describe the class of a for this equivalence relation. (3 pts.)

23. Let R/I denote the set of equivalence classes for the above equivalence relation. Show that the operations $[x] + [y] = [x + y]$ and $[x][y] = [xy]$ on R/I are well-defined and turn R/I into a ring (4 pts.). Show that the map π from R into R/I defined by $\pi(x) = [x]$ respects addition and multiplication. (2 pts.)

24. Let I be an ideal of R and let $I \subseteq J \lhd R$. Let $J/I = \{[x] : x \in J\}$. Show that $J/I \lhd R/I$. (5 pts.) Conversely show that any ideal of R/I is of the form J/I for some ideal J of R containing I. (6 pts.)

25. Show that the map φ from the set of ideals of R containing I into the set of ideals of R/I defined by $\varphi(J) = J/I$ is a bijection respecting inclusion. (4 pts.)

26. Show that an ideal I of R is maximal if and only if the ring R/I is a field. (6 pts.)